Deviations from Exponential Decay Law in the Time Evolution of Quantum Resonant States Described by Lorentzian Line Shape Spectral Distributions

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Abstract. This paper investigates the deviations from exponential decay law for quantum resonant states which can be approximately described by Lorentzian line shape spectral distributions. We point the significance of the Lorentzian distribution in both classical and quantum theory of resonances and its close relevance to the exponential decay law. Using quite general physical arguments, such as the finite expectation value of the energy and the kinematical dependence of the distribution, we investigate the appearance of these deviations for short and long times respectively. We construct an analogous to the continuity equation describing the correlation between exponential and non exponential decay. When a measuring perturbation cancelling the terms in the second part of the equation is possible, interesting questions arise as is for example whether the quantum Zeno effect, in the limit of very short times, does really appear. It is found that besides the homogeneity of the proposed continuity equation, other factors, such as the energy dependence of the resonance's complex energy shift, play an important role in the observability of the non exponential decay.

Keywords: resonance, decay width, energy shift, survival amplitude,non exponential decay, spectral distribution, continuity equation, exponential source term, Langevin equation, observability, quantum measurement, quantum Zeno effect.

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1. INTRODUCTION

Resonances in quantum mechanics correspond to the unstable quantum states, and acquire a complex energy spectrum. The latter seems to contradict with the structure of quantum mechanics which is built in terms of eigenvectors in Hilbert space supporting a real energy spectrum. However the unstable quantum state is an important example of irreversible phenomena in nature and has a distinct role in quantum mechanics, ranging from excited atomic states to short-lived elementary particles.

The resonant states interact and finally decay into continuum spectra. Their spectrum turns to be complex since the imaginary part of each pole, equal to Γ/2, is directly related to the mean lifetime of the corresponding state via the $\tau = \hbar / \Gamma$, and expresses half the energy width of the resonance. The real part is constituted by both the energy value of the unperturbed state and the energy shift due to the interaction with the continuum, giving the energy position of the resonance E_r, (or resonance mass in the relativistic case).The study of the decay of an unstable quantum state began with Gamow's theory [1] of alpha decay of atomic nuclei and Dirac's theory [2] of spontaneous emission of radiation by excited atoms, while a general treatment of decaying systems was given by Weisskopf and Wigner [3] and by Breit and Wigner [4]. Siegert [5] was the first to associate the complex poles in the S-matrix of Wheeler [6] to quantum resonances.

The signature of a resonant state is its spectral distribution. During the ages many models have been proposed for the choice of the spectral distribution, see for example **[7-12]** and references therein. Real and complex spectral distributions construct propagator functions with substantially different properties and conseque-nces on the system's time evolution. One of them seems to be the different type of non exponential decay for both regions of short and long times. The term non exponential decay is used for the description of the deviation from the exponential decay law, related to the evolution of the survival probability during an irreversible process. Although the exponential decay law is the universal hallmark of unstable states, deviations from it often prove to be more consistent with quantum mechanics. The dimensionless ratio

$$2E_r/\Gamma \equiv \beta \tag{1}$$

which is defined as twice the ratio of the energy position to the energy width of the resonance, is proved to be a very crucial and important quantity related to the appearance of such deviations, [13]. Deviations from the exponential law are present at times very close to the initial preparation time t = 0 and at very late times, while at "intermediate" times the exponential law represents a very good approximation. The intermediate – time region alone satisfies the simple composition law of probabi-lities $P(t_1)P(t_2) = P(t_1 + t_2)$. In this domain, therefore, a classical probability law operates, and the results for the two – step measurement are the same as for the one step measurement. At late times the decay law follows a power-law, which is however very difficult to observe experimentally because it occurs at times for which the survival probability is already vanishingly small. On the other hand, the deviations at small times occur within a very short time scale, for instance 10^{-15}s for the electromagnetic decays of an excited hydrogen atom [14] and even shorter for hadronic decays [15]. Beyond the theoretical prediction of such deviations there is much clear evidence for their experimental observation as well. The above may take place in many different branches of natural sciences, such as Nuclear Physics and Radioactivity, [16,17], Quantum field theory, [18], Atomic and Molecular Physics, [19,20], Charge transport, [21], Fluid dynamics, [22], Magnetism and spin dynamics, [16,23,24], Optics, [25], Chemical reactivity, [26], Biology, [27], Acoustics, [28], Geophysics, [24],Stochastic differential equations, [29], and may correspond to the presence of an unusual property in the system's dynamics, [30,31]. The above appear to be only a part of the extensive literature related to this subject.

This work studies the deviations from exponential decay law, in the framework of Lorentzian line shape spectral distributions. For this we first show the way the Lorentzian distribution appears in both quantum and classical mechanics. In quantum theory it is the work of Breit and Wigner [4], who studied the behavior of unstable particles, that revealed the Lorentzian distribution as the expression of the averaged phase shift of a wave in a scattering process. In classical mechanics the Lorentzian distribution describes the mean amount of energy absorbed per unit time, which is twice the mean value of the dissipative function, of a harmonically driven, harmonic oscillator with friction. It corresponds to a kind of dependence which is called dispersion-type frequency dependence of the absorption, [32]. We distinguish two types of spectral distributions, the Lorentzian with a semibounded spectrum, which we call truncated

Lorentzian distribution, and the generated complex Lorentzian distribution which occurs when keeping only the physical appropriate energy pole.

Next, we explore the appearance of the deviations from exponential decay in the limit of both short and long times. We use general arguments to show, that at short times the non decay probability falls off less rapidly than would be expected on the basis of the exponential decay law. The central point is that the time derivative of the survival amplitude at *t=0* is both finite and purely imaginary. In the limit of long times we use kinematical arguments to show that the decay has a time power law.

In the last part of the paper, we study the observability of the non exponential decay in the limit of very short times. For this we develop a model for the study of the correlation between exponential and non exponential decay, by constructing an analogous to the continuity equation. The homogeneity of this equation is achieved through the process of a measuring perturbation and is related to the observability of the non exponential decay. We discuss the possibility of the appearance of the quantum Zeno effect, in terms of a measuring process. Since the energy dependence of the resonance complex energy shift turns to be quite important, we explore these topics for various strengths of the above mentioned dependence.

2. LORENTZ LINE SHAPE DISTRIBUTION IN CLASSICAL AND QUANTUM MECHANICS

The fingerprints of a resonance reflect on its spectral distribution. The latter depends on the background and kinematical factors, and so we can only recover the centre of the resonance peak, which is the energy position of the resonance, and its width, defined as the energy distance between the points of half maximum of the distribution. However it is desirable to extract naturally the above mentioned quantities, as some kind of spectral information. This is done in absolute degree by the Lorentzian distribution, based on the Breit Wigner approximation,[4]. Breit and Wigner put the origins of the theory of quantum resonances by studying the behavior of unstable particles. They postulated that if an ustable particle at energy E_o decays according to the exponential decay law then the energy density should be approximately distributed according to the Breit-Wigner distribution which is Lorentzian line shape. This was done in the mathematical content of a scattering experiment which is captured by the scattering matrix and expressions derived from it. One of them, the scattering phase, measures the averaged phase shift which a wave experiences while passing through the scatterer, and according to the Breit-Wigner theory, should have an expression similar to the Lorentzian distribution. The Lorentzian spectral distribution is mathematically given by the following expression

$$f(E) = \frac{1}{\pi}\frac{\Gamma/2}{\left(E-E_r\right)^2+\Gamma^2/4} \qquad (2)$$

where $E_r = E_o + \Delta$ is the energy position of the resonance and Γ is the width. In this formalism E_o corresponds to the energy of the unperturbed state in which the system is initially prepared at *t=0* and Δ is the energy shift due to interaction with the continuum. It is easy to see that the Lorenzian distribution is the Fourier transform of the exponential factor of the form $e^{-iE_r t-\Gamma|t|/2}$, where we have used the atomic system of units $\hbar = 1$,with contributions from both the negative and positive time. Indeed we obtain for $-\infty < E < \infty$

$$f(E) \equiv \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt e^{-iE_r t-\Gamma|t|/2} e^{iEt} = \frac{1}{2\pi}\frac{1}{\Gamma/2+i(E-E_r)} + \frac{1}{2\pi}\frac{1}{\Gamma/2-i(E-E_r)}$$
$$= \frac{1}{\pi}\frac{\Gamma/2}{(E-E_r)^2+\Gamma^2/4}$$

(3)

We recognize two isolated poles at $E = E_r \pm i\Gamma/2$, that dominate the two pieces of the analytic Fourier transform. The one piece varies as $e^{-\Gamma t/2}$ for positive time and zero for negative time, while the other piece varies as $e^{\Gamma t/2}$ for negative time and zero for positive time. However neither piece corresponds to a compact autonomous state, since the state appears to be either created at *t=0*, or destroyed at *t=0*.

In classical mechanics the Lorentz line shape arises in the problem of a harmonically driven, harmonic oscillator with friction. The differential equation describing its motion is the following

$$\ddot{x} + \gamma\dot{x} + \omega_o{}^2 x = \frac{f}{m} e^{-i\omega t} \tag{4}$$

where γ stands for the damping constant, ω_o is the frequency of the unperturbed problem, *f* is the amplitude of the force, *m* is the mass of the system, and ω is the driving frequency. It is to be understood that we take in account only the real part of the right-hand side. It is easy to see that a particular solution of the above equation has the form

$$x(t) = \frac{f/m}{\omega_o{}^2 - \omega^2 - i\omega\gamma} e^{-i\omega t} \tag{5}$$

It is then interesting to calculate the two point correlation function coming as

$$\langle x^*(0)x(t)\rangle \equiv \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega \frac{f^*/m}{\omega_o{}^2-\omega^2+i\omega\gamma}\cdot\frac{f/m}{\omega_o{}^2-\omega^2-i\omega\gamma} e^{-i\omega t} \tag{6}$$

The correlation function may be evaluated by complex contour integration yielding

for positive and negative times,

$$\langle x^*(0)x(t)\rangle = \begin{cases} \left|\frac{f}{m}\right|^2 \left\{ \frac{e^{-\gamma t/2}e^{-i\Omega t}}{2\Omega\gamma(2\Omega - i\gamma)} + \frac{e^{-\gamma t/2}e^{i\Omega t}}{2\Omega\gamma(2\Omega + i\gamma)} \right\} & for\ t>0 \\ \left|\frac{f}{m}\right|^2 \left\{ \frac{e^{\gamma t/2}e^{-i\Omega t}}{2\Omega\gamma(2\Omega + i\gamma)} + \frac{e^{\gamma t/2}e^{i\Omega t}}{2\Omega\gamma(2\Omega - i\gamma)} \right\} & for\ t<0 \end{cases} \tag{7}$$

or
$$\langle x^*(0)x(t)\rangle = \begin{cases} \left|\frac{f}{m}\right|^2 \left\{ \frac{2e^{-\gamma t/2}}{\gamma(4\Omega^2+\gamma^2)}\cos\Omega t + \frac{e^{-\gamma t/2}}{\Omega(4\Omega^2+\gamma^2)}\sin\Omega t \right\} & for\ t>0 \\ \left|\frac{f}{m}\right|^2 \left\{ \frac{2e^{\gamma t/2}}{\gamma(4\Omega^2+\gamma^2)}cos\Omega t + \frac{e^{\gamma t/2}}{\Omega(4\Omega^2+\gamma^2)}\sin\Omega t \right\} & for\ t<0 \end{cases} \tag{8}$$

where we have defined the frequency $\Omega = \sqrt{\omega_o^2 - \gamma^2/4}$. If the approximation $\omega_o >> \gamma$ is used then the above formulae take the much simpler form

$$\left\langle x^*(0)x(t)\right\rangle \cong \left|\frac{f}{m}\right|^2 \frac{e^{-\gamma|t|/2}}{2\gamma\omega_o^2}\cos\omega_o t \tag{9}$$

We see that the temporal behavior of the correlation function is exponentially damped for both positive and negative time and thus directly related to the Lorentzian spectral distribution as we have seen earlier.

We can attack to the problem in a different way. Let us write $\frac{f/m}{\omega_o^2 - \omega^2 - i\omega\gamma} = \rho e^{i\vartheta}$ and find $\rho(\omega,\gamma) = \frac{f}{m}\frac{1}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \omega^2\gamma^2}}$ and $\tan\vartheta = \frac{\omega\gamma}{\omega_o^2 - \omega^2}$. In this way a particular integral of equation (4) is given by $x(t) = \rho e^{i(\vartheta - \omega t)}$. The general solution of (4) with zero on the right –hand side is given by

$$x(t) = ce^{-\gamma t/2}e^{i\Omega t} + de^{-\gamma t/2}e^{-i\Omega t} = ae^{-\gamma t/2}\cos(\Omega t + \varphi) \tag{10}$$

In this way the solution of (4) is the sum

$$x(t) = ae^{-\gamma t/2}\cos(\Omega t + \varphi) + \rho(\omega,\gamma)\cos(\omega t - \vartheta) \tag{11}$$

Since the first term decreases exponentially with time, after a sufficient time only the second term survives. In fact the first term describes the transient behavior of the system. In contrast with a resonance without damping, the amplitude of oscillation, quantity $\rho(\omega,\gamma)$, depends on the frequency of the driving force, and acquires its maximum value when $\omega = \sqrt{\omega_o^2 - \gamma^2/2}$. However this maximum is not infinite as for the case of the resonance without friction. If the damping constant is small enough then the range of resonance is very close to ω_o. Let us again assume that $\omega_o >> \gamma$, and write $\omega = \omega_o + \delta\omega$, with $\delta\omega$ very small. We will then find that

$$\rho(\omega,\gamma) = \frac{f}{m}\frac{1}{2\omega_o\sqrt{\delta\omega^2 + \gamma^2/4}} \tag{12}$$

During the oscillation the system continuously absorbs energy from the source of the external force, which in turn dissipates to the environment. The mean amount of energy absorbed per unit time is given as twice the mean value of the dissipative function, [33]. The latter is in generally a quadratic function of the $\dot{x}_i$ for a system with many degrees of freedom, and in our case is given by $F = \frac{1}{2}\gamma m\dot{x}^2$. It is easy to see that

$$F = \frac{1}{2}\gamma m \rho^2(\omega,\gamma)\sin^2(\omega t - \vartheta)\omega^2 \Rightarrow \overline{F} = \frac{1}{4}\gamma m \omega^2 \left(\frac{f}{m}\right)^2 \frac{1}{4\omega_o{}^2\left(\delta\omega^2 + \gamma^2/4\right)} \Rightarrow$$
$$\overline{F} \cong \frac{f^2}{4m}\frac{\gamma/4}{\left(\delta\omega^2 + \gamma^2/4\right)} \tag{13}$$

Clearly we have reached to the Lorentzian line shape for the energy absorption, and this kind of dependence is called dispersion-type frequency dependence of the absorption, **[32]**. Thus, classical and quantum physics include phenomena that they are supported by the Lorentzian line shape distribution.

3. DEVIATIONS FROM EXPONENTIAL DECAY LAW IN THE LIMIT OF SHORT AND LONG TIMES.

In the previous section we used the correlation function discussing resonances in classical mechanics. The quantum analogous of the two point correlation function is the survival amplitude, defined as

$$a(t) \ \equiv \ \left\langle \Phi_O \middle| e^{-iHt} \middle| \Phi_O \right\rangle = \left\langle \Phi_O \middle| \Phi(t) \right\rangle \tag{14}$$

Here $\left|\Phi_O\right\rangle$ represents the initial state of the system, meaning the state in which it has been prepared at $t=0$ and H is the system's Hamiltonian. The latter is assumed to be exactly known and usually corresponding to the unperturbed problem where no interaction to the continuum is possible. It can be described by a localized wave packet whose energy E_O is inside the continuous spectrum. In this way $\left|\Phi(t)\right\rangle$ represents the time evolution of the initial state at arbitrary times. We can now insert the unit operator $\hat{I}$ constructed by the complete set of states of the Hamiltonian

$$\hat{\mathrm{I}} = \int dE \left|E\right\rangle\left\langle E\right| \tag{15}$$

which obviously satisfy the Schrödinger equation $\hat{H}\left|E\right\rangle = E\left|E\right\rangle$, and get

$$a(t) = \left\langle \Phi_O \middle| \mathrm{e}^{-iHt} \middle| \Phi_O \right\rangle \Rightarrow a(t) = \left\langle \Phi_O \middle| \int dE \left|E\right\rangle\left\langle E\right| \mathrm{e}^{-iHt} \middle| \Phi_O \right\rangle \Rightarrow$$
$$a(t) = \left\langle \Phi_O \middle| \int dE \left|E\right\rangle\left\langle E\right| \mathrm{e}^{-iEt} \middle| \Phi_O \right\rangle = \int dE\, \mathrm{e}^{-iEt} \left|\left\langle \Phi_O \middle| E \right\rangle\right|^2 \tag{16}$$

where again we use the atomic system of units where $\hbar = 1$. The last term of the above equation defines the spectral distribution in terms of the Hamiltonian states and the initial state as well, through the $\rho(E) = \left|\left\langle \Phi_O \middle| E \right\rangle\right|^2$. It demonstrates the physical meaning of the spectral distribution as the weight function for contribution of each of the energy states of the Hamiltonian to the construction of the survival amplitude. In other words the following quantity

$$\int_E^{E+dE} \rho(\varepsilon)d\varepsilon \tag{17}$$

is the probability that the energy of the state lies in the interval [*E*,*E*+*dE*]. It is clear from (16) that the Fourier transform of the spectral distribution is equal to the survival amplitude.The survival probability is given by

$$P(t) = \left|\langle \Psi_o | \Phi(t) \rangle\right|^2 = |a(t)|^2 \qquad (18)$$

and must satisfy the following two conditions: $P(0)=1$, and $P(\infty)=0$, due to preparation and non stationarity respectively. In order to compute the survival amplitude $a(t)$,we must know $|\Phi(t)\rangle$, which is the solution of the time dependent Schrödinger equation. This is not trivial at all since it demands the knowledge of all of the interactions in a generally complicated problem. Alternatively we can assume a specific form for the spectral distribution of the system, based on certain properties and conditions that have to be fulfilled, and then calculate the survival amplitude through Fourier or Laplace transforms, [7-10,13], (and references therein). Among the various types of spectral distributions the Lorentzian, although still an approxima-tion is rather the most popular since not only occurs in many branches of physics and different phenomena: deexcitation of atomic levels, alpha decay, resonant scattering, but also because it constitutes the generator of many other types of distributions or extensions thereof, [13,35-43]. More than this it has been seen that the Lorentzian spectral distribution can be directly related to Gamow vectors and exponential time evolution without violating causality, [35]. However an issue to be discussed has to do with the violation of its spectrum boundedness. In other words the Lorentzian spectral distribution violates the spectral condition to obtain a strict exponential decay. This happens because we admit states with arbitrarily large negative energies. This violation would also violate the second law of thermodynamics, [34] and the uncertainty principle as well. For the first case we can imagine suitable interactions to take arbitrarily large amounts of energy from the system. The first law of thermodynamics can be satisfied and yet the available energy from the system is arbitrarily large. This must not be possible and so the unbounded spectrum by itself should not occur. For the second case we have to think that the infinite negative potential energy, would confine a particle in a very small area, for example an electron near the nucleus. In this way both *δx* and *δp* tend to zero which is a contradiction to the uncertainty principle. So, since the distribution should actually be truncated to positive *E*, corresponding to the threshold of the continuum spectrum, we equivalently set $\rho(E)=0$ for $E \le 0$. The $\rho(E)$ is called *Truncated Lorentzian Distribution TLD,* and its general form is the:

$$\begin{aligned} \rho^{TLD}(E) &= N \frac{\Gamma(E)/2}{\left(E - E_o - \Delta(E)\right)^2 + \Gamma(E)^2/4} \quad \text{for } E > 0 \\ &= 0 \text{ for } E < 0 \end{aligned}, \qquad (19)$$

where $\Delta(E)$ is the energy shift, $\Gamma(E)$ is the energy width of the resonance and N is a normalization factor different from 1/π since the distribution is now semibounded. For the same reason both the energy shift and the energy width are now energy dependent. The distribution $\rho^{TLD}(E)$ is a real function of energy and must satisfy the following condition of normalization, since probability is conserved:

$$\int_0^\infty \rho^{TLD}(E)\, dE = 1 \qquad (20)$$

which is equivalent to write $a(0) = a^*(0) = 1$. This equation determines the value of quantity *N* and gives:

$$N\left(\beta(E),\delta(E)\right) = \left\{\left[\delta(E) + \tan^{-1}\beta(E)\right]\right\}^{-1} \tag{21}$$

where quantity $\delta(E)$ represents an angle close but not equal to $\pi/2$ as a consequence of the energy dependence of both quantities $\Delta(E)$ and $\Gamma(E)$, analytically shown in **[13].** In fact the energy dependence of the angle *δ* makes the Lorentzian a better approximation. It is quite obvious from eq. (19) that the truncated Lorentzian distribution reveals two complex energy poles that come as the complex conjugates of each other:

$$z = \mathrm{E}_r(E) + i\Gamma(E)/2 \quad , \quad z^* = \mathrm{E}_r(E) - i\Gamma(E)/2 \tag{22}$$

However since the propagator function must properly describe the irreversible time evolution of the system, only the z^* pole should be chosen in order to give the correct exponential decay law of the form $P(t) \sim e^{-\Gamma t/\hbar}$. Having this in mind some-one can propose instead a complex function of energy that intrinsically carries irreversibility and causality, and arises from $\rho^{\mathrm{T}LD}(E)$ by keeping only the z^* pole, meaning the Complex Lorentzian Distribution *CLD*:

$$\rho^{CLD}(E) = \frac{\tilde{N}\left(\beta(E)\right)}{2} \frac{i}{E - E_r(E) + i\,\Gamma(E)/2} \tag{23}$$

leading to a time evolution of the form $\left|\Phi(t)\right\rangle = \theta(t)e^{-iE_r t-(1/2)\Gamma t}\left|\Phi_o\right\rangle$.

The Paley-Wiener theorem **[44]** states that if the spectrum is bounded from below, then the survival amplitude and hence the survival probability decreases to zero as time passes less rapidly than any exponential function, and thus deviates from expo-nential decay. In fact we can show that deviations from exponential decay in the limit of long times, arise from clearly kinematical arguments. As we have already pointed, the spectral distribution depends on the background and kinematical factors. For example we can separate the phase space factor *σ(E)* in the spectral distribution and write the latter as

$$\rho(\varepsilon) = f(\varepsilon)\cdot\sigma(\varepsilon) \tag{24}$$

where the form factor *f(ε)* expresses the energy distribution of the decay products, to whom the unstable state is finally distributed. As time grows the wave packet of the initial state spreads so that the decay products separate sufficiently far to be outside each other's influence, and the distribution becomes clearly kinematic. This means that the form factor is smoothly varying after some large but finite time, since the corresponding interactions between the decay products become negligible. The remaining phase space factor has the form

$$\sigma(E) = \frac{d}{dE}\left(\frac{4}{3}\pi k^3\right) = 4\pi k^2 \frac{dk}{dE} \tag{25}$$

where k is the wave number associated to the remaining kinetic energy via the $E = \frac{k^2}{2m}$. In this way we have

$$\sigma(E) = 8m\pi E \cdot \sqrt{\frac{m}{2E}} = 4\sqrt{2}\pi m^{3/2}\sqrt{E} \tag{26}$$

Thus, the survival amplitude for large times behaves as

$$\begin{aligned} a(t) &= \int_0^\infty dE e^{-iEt}\rho(E) \cong \int_0^\infty dE e^{-iEt}\sigma(E) = 4\sqrt{2}\pi m^{3/2}\int_0^\infty dE e^{-iEt}\sqrt{E} \\ &= \frac{1}{t}\frac{1}{\sqrt{t}}4\sqrt{2}\pi m^{3/2}\int_0^\infty dz e^{-iz}\sqrt{z} \sim t^{-3/2} \end{aligned} \tag{27}$$

where we have reached the same result as in **[13]** but in a different way.

We may now examine the behavior of the survival amplitude, in the limit of very short times. The spectrum of the Hamiltonian is semibounded and in addition we assume that the expectation value of the energy at t=0 is finite. The expectation value of the energy may be taken as $\left\langle \Phi_O \middle| \hat{\mathrm{H}} \middle| \Phi_O \right\rangle$ or alternatively via the use of the spectral distribution,

$$\left\langle E \right\rangle = \int_0^\infty \varepsilon\rho(\varepsilon)d\varepsilon \tag{28}$$

It is interesting to notice that the time derivative of the survival amplitude at *t=0* is connected to the expectation value of the energy since it is true that

$$\dot{\alpha}(t) = \int_0^\infty dE(-iE)e^{-iEt}\rho(E) \Rightarrow \left|\dot{\alpha}(0)\right| = \int_0^\infty E\rho(E)dE = \left\langle E \right\rangle \tag{29}$$

Writing down the survival probability we can actually write

$$P(t) = \alpha(t)\cdot\alpha^*(t) \Rightarrow \frac{dP(t)}{dt} = \dot{\alpha}(t)\cdot\alpha^*(t) + \alpha(t)\cdot\dot{\alpha}^*(t) \tag{30}$$

The time derivative of the survival amplitude is continuous since we have

$\left|\dot{\alpha}(t)\right| = \left|\int_0^\infty dE(-iE)e^{-iEt}\rho(E)\right| \le \int_0^\infty |E||\rho(E)|dE < \infty$ and $\left\langle E \right\rangle$ is finite. It is easy to see that quantity $\dot{\alpha}(0)$ is a purely imaginary quantity equal to

$$\dot{\alpha}(0) = -i\left\langle E \right\rangle \tag{31}$$

If this is the case then the time derivate of the survival probability at t=0, is equal to

$$\left.\frac{dP(t)}{dt}\right|_{t=0} = \dot{\alpha}(0)\cdot\alpha^*(0) + \alpha(0)\cdot\dot{\alpha}^*(0) = \dot{\alpha}(0)\cdot 1 + 1\cdot\left(-\dot{\alpha}(0)\right) = 0 \tag{32}$$

The last result shows that the decay can not be exponential at very short times since then we should actually have, $\left.\frac{dP(t)}{dt}\right|_{t=0} = -\Gamma$. So at sufficient small time, the non decay probability falls off less rapidly than would be expected on the basis of the exponential decay law.

For each case of distribution, *TLD* or *CLD*, it was shown in **[13]** that the time evolution of an unstable system is constituted by two parts: the exponential decay part and the non exponential decay part. The exponential decay parts were exactly the same for the two distributions, while the non exponential decay parts had substantial in between differences. These differences had mainly to do with the form of the non exponential survival amplitude. The real spectral distribution gave the following amplitude:

$$\mathrm{I}^{TLD}(\beta,\alpha) = N(\beta,\delta) \int_{\delta}^{\tan^{-1}\beta} \exp[i\alpha(\beta-\tan\theta)]\, d\theta \tag{33}$$

while the complex distribution gave:

$$I^{CLD}(\beta,\alpha) = \frac{N(\beta,\delta)}{2} \int_{\delta}^{\tan^{-1}\beta} \mathrm{d}\theta \exp\{i[\alpha(\tan\theta-\beta)-\theta]\}\, /\cos\theta \tag{34}$$

In the above relations we have used the following dimensionless quantities, $\alpha \equiv t/2\tau_d$ where τ_d is the mean lifetime of the resonant state and $\beta \equiv 2E_r/\Gamma$ as twice the ratio of the energy position to the decay width of the resonant state. In addition the non exponential amplitudes satisfy a different differential equation. Both of them come as a classical Langevin type of equation, [45], with quantity $\omega(\beta,\delta) \equiv (\beta - \tan\delta)$ (35)

corresponding to the frequency of the rapidly oscillating stochastic terms, and carrying information from the limits of the spectrum, [13,46].

4. CORRELATION BETWEEN EXPONENTIAL AND NON EXPONENTIAL AMPLITUDE.

The exponential part of the decay is totally determined by the width function $\Gamma(E)$ since the latter describes the rate of exponential decay of the system's survival probability. This quantity is produced through the interaction of the system with the continuum, and there exist many different methods for its calculation. For example it can be constructed through Fermi's golden rule, [47], where it is produced from the contribution of the matrix elements of the interaction potential, or alternatively through path integral methods, [48], and contributions of the classical action inside the potential barrier. The energy shift $\Delta(E)$ is also produced by the interaction terms and according to [48] is given as a function of the derivative of the classical action inside the potential barrier with respect to energy. It is clear from the above that both quantities $\Gamma(E)$ and $E_r(E)$ are similarly constructed and depend on the type and strength of the interaction with the continuum, and so does their ratio β.

Following Dirac's formalism we can define a vector whose energy wave function is a Lorentzian distribution, meaning the

$$|\phi\rangle = \int dE\,|E\rangle\langle E|\phi\rangle = \int dE\,|E\rangle\left(i\sqrt{\frac{\Gamma}{2\pi}}\frac{1}{E-(E_r - i\Gamma/2)}\right) \tag{36}$$

If we choose the boundaries of integration to be $-\infty \le E < \infty$,the length of this vector is given by

$$\langle\phi|\phi\rangle = \int_{-\infty}^{\infty} dE\langle\Psi^G|E\rangle\langle E|\Psi^G\rangle = \int_{-\infty}^{\infty} dE\frac{\Gamma}{2\pi}\frac{1}{(E-E_r)^2+\Gamma^2 4} = 1 \tag{37}$$

However if we choose the boundaries according to the rules of standard quantum mechanics, $0 \le E < \infty$, we get

$$\langle\phi|\phi\rangle = \int_0^\infty dE \langle\Psi^G|E\rangle\langle E|\Psi^G\rangle = \int_0^\infty dE \frac{\Gamma}{2\pi}\frac{1}{(E-E_r)^2+\Gamma^2 4} = \frac{1}{\pi}\int_{-\beta}^\infty dz \frac{1}{z^2+1} = 1 - \frac{1}{\pi}\left\{\beta^{-1} - \frac{\beta^{-3}}{3} + ..\right\} \quad (38)$$

The first length equals 1, and corresponds to the pure exponential decay law as we have already seen in eq. (3). The second length is less than 1, and corresponds to the mixed propagation, exponential and non exponential, given as a function of the ratio β. It was shown in [13] and can be established once more from the above relations, that in the limit where β tends to infinity the exponential decay becomes the only contribution in the system's time evolution, (the results of equations 37 and 38 coincide). On the contrary when β takes values close to 1 the exponential decay part becomes much less significant. For intermediate values of the ratio β we get an interplay between exponential and non exponential decay and this is reflected on the values of the length of eq. (38). It is easily seen that equation (38) describes an increasing function of β. So there is an one by one correspondence between the strength of the exponential decay and the values of the ratio β. In this way the correlation between the exponential and the non exponential decay generates translations in the β parameter and we may think of it as the generator of the one dimensional space introduced by β. Inside this generalized space, the non exponential part of the decay will classically flow, while the space points may act as secondary either destructive or constructive sources. The study of this kind of behaviour is expected to extract information about the correlation between the two types of evolution.

The continuity equation expresses the fact that inside a finite volume "mass" or "charge" is conserved in the absence of external sources and this makes the equation homogeneous. As a first step in the mathematical analysis that follows a generalized density and current must be properly introduced. Classical intuition is related to probabilities which are the directly observed quantities. But probabilities do not propagate. Propagation is for the amplitude. Thinking so, the generalized density is defined by the non exponential amplitude itself, meaning $\mathrm{A} = \mathrm{I}^{TLD}(\beta,\alpha)$ or $\mathrm{I}^{CLD}(\beta,\alpha)$ and depends on both space, (meaning $\beta \equiv 2E_r/\Gamma$) and time, (meaning $\alpha \equiv t/2\tau_d$). The corresponding current comes then naturally as

$$J \equiv \beta \frac{\partial \mathrm{A}(\beta,\alpha)}{\partial \alpha} \quad (39)$$

The continuity equation comes as [49],

$$\frac{\partial \rho}{\partial t} + div\,\vec{J} = 0 \quad (40)$$

and takes the following form for the case of the non exponential amplitudes:

$$\frac{\partial \mathrm{A}(\beta,\alpha)}{\partial \alpha} + \frac{\partial}{\partial \beta}\left(\beta \frac{\partial \mathrm{A}(\beta,\alpha)}{\partial \alpha}\right) = 0 \quad (41)$$

In the following paragraphs the specific form of eq. (41) is investigated for the cases of real and complex spectral distributions that were previously mentioned. It is easy to see after a little piece of algebra, that the conservation law takes the following form:

$$\frac{\partial \mathrm{I}^{TLD\ or\ CLD}(\beta,\alpha)}{\partial \alpha} + \frac{\partial}{\partial \beta}(\beta \frac{\partial \mathrm{I}^{TLD}(\beta,\alpha)}{\partial \alpha}) = [\frac{\mathrm{sech}(\ln\beta)}{4N(\beta,\delta)} - 1]\times$$

$$\frac{iN(\beta,\delta)}{\tau_d} \int_{\delta}^{\tan^{-1}\beta} \left[(\beta\ or\ 0.5) - (1\ or\ 0.5i)\tan\theta\right]\exp[i\alpha(\tan\theta - \beta)]\, d\theta$$

(42)

It is obvious that the continuity equation is inhomogeneous since an additional term in the right hand of the equation appears and which we define as the exponential source term, (*est*). However the lack of homogeneity has important consequences in terms of physics. Due to the first term of the left hand side of eq. (42), the *est* corresponds to a first time derivative of an amplitude and can be cancelled if a suitable measuring perturbation is applied. In the limit of short times this cancellation would make the non exponential decay uncorrelated to the exponential one, and consequently observable. So the question is whether these terms can be cancelled at this regime of time. This can be achieved only if the *est* takes finite values which in turn ensures that we can actually follow the system through its evolution. We are interested in the limit of short times not only because it is the regime of time where the non exponential decay becomes important but also because we may want to discuss some interesting and peculiar phenomena such as the quantum Zeno effect.

The *est* term comes as a function of quantities α and β, and of the resonance's lifetime τ_d and corresponds to the correlation between exponential and non-exponential decay during the system's time evolution. The first thing to notice is the fact that the continuity equation becomes homogeneous when the following condition is satisfied:

$$4N(\beta_o, \pi/2) = \mathrm{sech}(\ln\beta_o) \qquad (43)$$

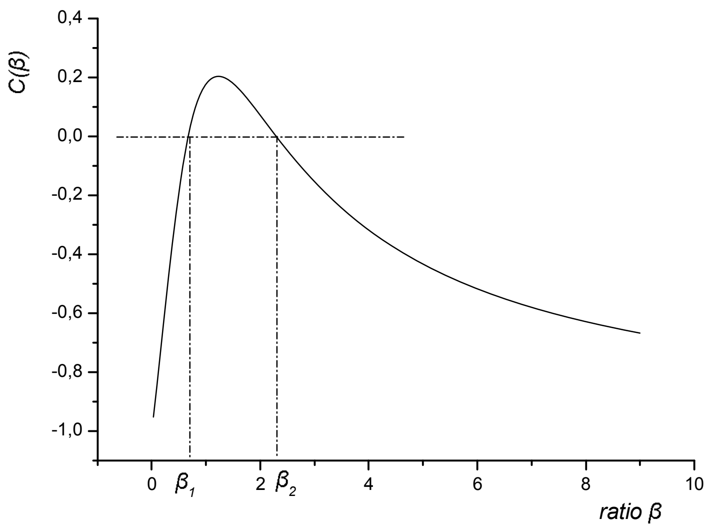


FIGURE 1. The variation of quantity *C(β)* with *β*. Its Lorentzian shape shows a maximum at *β=1.25*, and approaches -1 for both $\beta \to 0, \infty$.

For a given system, the different possible values of quantity β correspond to different kind and strength of interactions with the continuum.By solving numerically the

transcendental equation (43) we find two solutions namely $\beta_1 \cong 0.67$ and $\beta_2 \cong 2.29$. These answer directly to the question of how close to unit must ratio *β* be in order for the non-exponential decay to dominate the evolution. On the other hand and since *β* also appears in the limit of integration in the right term of (42), as $\beta \to \infty$ the flow disappears and the same of course happens with the correlation between the two kinds of evolution. In fact we can draw in Figure 1 that follows quantity *C(β)* as a function of *β* defined by

$$\mathrm{C}(\beta)=[\frac{\operatorname{sech}(\ln\beta)}{4N(\beta,\pi/2)}-1] \tag{44}$$

For both regions of $\beta \to 0$ and $\beta \to \infty$,*C(β)* approaches -1. It is interesting to notice that *C(β)* has itself a Lorentzian-like shape. When condition (43) is satisfied, the non exponential part of the decay is conserved and the interaction between the two different types of propagation is negligible.

However we are interested for other values of the ratio β as well. For this we exa-mine in more details the behavior of the *est* term, in the lines that follow. Concernig the real spectral distribution, we first treat the case where the interaction with the continuum, through a potential barrier for example, changes both *δ(E)* and *β(E)*in a way that the following relation holds

$$\beta(E) \cong \tan\delta(E) \tag{45}$$

This would correspond to the situation where the complex energy shift is strongly energy dependent. This makes *δ(E)* also strongly energy dependent and reflects on the magnitude of its derivative. We can then approximate *est* ^{R} and after a little piece of algebra find that

$$est^R{}_{\left|\frac{d\delta}{dE}\right|>>} \cong i\frac{N(\beta,\delta)}{1+\beta^2}\left[\frac{\operatorname{sech}(\ln\beta)}{4N(\beta,\delta)}-1\right]\left(\tan\delta-\beta\right)^2 \tag{46}$$

It is interesting to notice that there is no time dependence of the *est* ^{R}, to be discussed in the next section. For all the other types of smooth dependent complex energy shifts we can approximate *est* ^{R} as follows

$$est^R{}_{t>} \cong \frac{i}{2\tau_d}\left[N(\beta,\delta)-\frac{\operatorname{sech}(\ln\beta)}{4}\right]\mathrm{x}$$

$$\left\{\begin{array}{l} -i\omega(\beta,\delta)\exp\left(i\omega(\beta,\delta)t\right)\sum_{n=0}^{\infty}\frac{n!}{\left(t/\tau_d\right)^{n+1}}\left(\frac{1}{\left(i\tan\delta+1\right)^{n+1}}-\frac{1}{\left(i\tan\delta-1\right)^{n+1}}\right)+ \\ \exp\left(i\omega(\beta,\delta)t\right)\sum_{n=0}^{\infty}\frac{(n+1)!}{\left(t/\tau_d\right)^{n+2}}\left(\frac{1}{\left(i\tan\delta-1\right)^{n+1}}-\frac{1}{\left(i\tan\delta+1\right)^{n+1}}\right)+ \\ \sum_{n=0}^{\infty}\frac{(n+1)!}{\left(t/\tau_d\right)^{n+2}}\left(\frac{1}{\left(i\beta+1\right)^{n+1}}-\frac{1}{\left(i\beta-1\right)^{n+1}}\right) \end{array}\right\} \tag{47}$$

where the above formula stands for times beyond the short time regime. In the region of very short times it is easy to approximate *est* ^{R} according to the following formula

$$est^{R}{}_{t<<} \cong \frac{1}{\tau_d}\left[\frac{\mathrm{sech}(\ln\beta)}{4} - N(\beta,\delta)\right] \mathrm{x}$$
$$\left\{ \begin{array}{l} i\left[\beta\left(\tan^{-1}\beta-\delta\right)+\ln\dfrac{\cos\tan^{-1}\beta}{\cos\delta}\right]- \\ \dfrac{t}{2\tau_d}\left[\left(\tan^{-1}\beta-\delta\right)\left(1-\beta^2\right)-2\beta\ln\dfrac{\cos\tan^{-1}\beta}{\cos\delta}+\tan\delta-\beta\right]\end{array}\right\} \quad (48)$$

In the case of the complex spectral distribution a different type of est^{C} is revealed, where c stands for the complex distribution. It is not difficult to find the following operator that connects the *est* for the two different types of spectral distribution

$$est^{C} \equiv \hat{C} \to R\ est^{R} = \frac{1}{2}\left(1-i\beta-\frac{\partial}{\partial\alpha}\right)est^{R} \quad (49)$$

We may now focus on the short time regime and discuss the observability of the non exponential part of the decay. The case of long times can easily be handled, at least in a theoretical base, since the polynomial form of the non exponential propagation always dominates the exponential part after a certain number of lifetimes [13]. In the latter case the only problem to be solved experimentally has mostly to do with the very small magnitude of the propagation at these limits of time, while in the former case there exist other difficulties as well, related to the quantum nature of the resonant states, and the tunneling process, [13,46,48]. The angle δ that enters the calculations as the second limit of integration in the propagator integral equations, carries information from the limits of the energy spectrum at very short times and contributes significantly to the non exponential propagation. An earlier study of these contributions, [46], concerned the question of observability "of early time departures from Fermi's golden rule" where the issue of the non exponential propagation at $t=0$ was discussed at length.

As we have already said *est* terms correspond to the first time derivative of a quantity with the same dimensions as those of the amplitude meaning a matrix element of the form $\left\langle \Phi_O \middle| \hat{\mu}(t) \middle| \Phi_O \right\rangle$, where $\hat{\mu}(t)$ stands for the perturbation effected by a measurement, and Φ_O stands for the initial state of the system at *t=0* with energy E_O as has been already mentioned. The evolution of the system is described by

$$i\hbar\frac{\partial}{\partial t}\left|\Phi(t)\right\rangle = \hat{H}\left|\Phi(t)\right\rangle \quad (50)$$

where $\hat{H}$ is the Hamiltonian operator and $\left|\Phi(0)\right\rangle = \Phi_O$. We assume that the measurement process at the limit of very short times permits interaction with the continuum. For example we may have a time dependent potential barrier that alters both the classical allowed and classical forbidden region of motion, and eventually quantities *δ* and *β*. This also changes the state of the system from Φ_O to $\left|\Phi(t)\right\rangle$, and makes quantity $\left\langle \Phi_O \middle| \Phi(t)\right\rangle = \left\langle \Phi_O \middle| \hat{\mu}(t) \middle| \Phi_O \right\rangle \equiv a(t)$ the survival amplitude. It follows directly from (50) that $i\hbar\dot{\alpha}(t) = \left\langle \Phi_O \middle| \hat{H} \middle| \Phi(t)\right\rangle$ and for *t=0* we have

$$i\hbar\dot{a}(0) = \left\langle \Phi_O \middle| \hat{H} \middle| \Phi_O \right\rangle = E_O \quad (51)$$

The above relation expresses the fact that in the limit of very short times quantity $\dot{\alpha}(0)$ must be purely imaginary. This is true for the real Lorentzian distribution, as can be easily proved from (48) where we get

$$\dot{a}(0) \cong i\frac{1}{\tau_d}\left[\frac{\operatorname{sech}(\ln\beta)}{4} - N(\beta)\right]\left[\beta\left(\tan^{-1}\beta - \delta\right) + \ln\frac{\cos\tan^{-1}\beta}{\cos\delta}\right] \qquad (52)$$

This is not true however for the complex distribution due to relation (49) which makes $\dot{\alpha}(0)$ a complex and not purely imaginary quantity. This means that the complex spectral distribution does not provide a complete theory for the observability of the non exponential decay in the limit of very short times. Let us focus on the purely imaginary character of quantity $\dot{a}(0)$. Following the steps that lead to exponential decay, we assume that the system is prepared in state $\left|\Phi_O\right\rangle$ at a time t and that the probability of decay in a time interval is Γdt. Thus if the probability that the system was in state $\left|\Phi_O\right\rangle$ is *P(t)* then the probability that the system will be in state $\left|\Phi_O\right\rangle$ at a time *t+dt* will be $P(t+dt) = P(t)(1-\Gamma dt)$ and this of course leads to exponential decay. If we now work with the survival amplitude we will take

$$P(t+dt) = \left|a(t+dt)\right|^2 = \left|a(t)+\dot{a}(t)dt\right|^2 = \left|a(t)\right|^2\left|1+\frac{\dot{a}(t)}{a(t)}dt\right|^2 = P(t)\left|1+\frac{\dot{a}(t)}{a(t)}dt\right|^2 \qquad (53)$$

Since $a(0)=1$ and $\dot{a}(0)=i\gamma$ we have for *t=0*, $P(dt)=1+\gamma^2 dt^2$. The latter is greater than unit however this inconsistency is removed if we take in account all the remaining derivatives of the amplitude in the expansion of $a(t+dt)$. The case of exponential decay gives $P(dt)=1-\Gamma dt$. If we look at $P(dt)$ as a polynomial of *dt* we note the absence of *dt* in the amplitude method. This is due to our mistake assumption in constructing the exponential dependence,that the survival probability changes entirely due to transitions out of the initial state where we have not taken in account transitions back to the initial state. The latter would cancel the $-\Gamma dt$ term.

On the other hand eq. (52) together with (31) serve as a tool for the calculation of the angle *δ(E)* that enters the Lorentzian distribution and takes in account the energy dependence of the complex energy shift. Indeed we get the transcendental equation

$$\frac{\langle E\rangle\tau_d}{C(\beta)} + \beta\tan^{-1}\beta + \ln\left(\cos\tan^{-1}\beta\right) \cong \beta\delta + \ln\cos\delta \qquad (54)$$

We can now discuss the finiteness of the time derivative of the survival amplitude. In fact we can divide the physical systems in two categories: these where the expecta-tion value of the energy is finite and these where it tends to infinite. Concerning the first case, we have already shown that when the expectation value of the energy is finite then the decay rate approaches zero as $t \to 0$.The time derivative of the amplitude expresses the rate with which the measuring perturbation should change in time in order for the non exponential decay to be observed. Combining the above we are left with the conclusion that in this case we can follow the system at sufficie-ntly small time and observe deviations from exponential propagation. But deviations from exponential decay are equivalent to say that at *t=0* the decay rate tends to zero and not equal to *–Γ*. Thus, if the unstable system is monitored for its existence at sufficiently small intervals of time, it would appear to be longer lived than if it were monitored at intermediate intervals, where the decay law is exponential. These are the conditions for the quantum Zeno's paradox to appear, which states that in the limit of continuous monitoring the system does not decay at all. The quantum Zeno effect was first understood by von Neumann, [50]. While analyzing the thermodyna-mic features of quantum ensembles von Neumann proved that any given state Φ_O of a quantum mechanical system can be “steered” into other state Ψ of the same Hilbert space, by performing a series of very frequent measurements. If Φ_O and Ψ coincide, the evolution is frozen, meaning that a quantum Zeno effect takes place. The classi-cal allusion to the sophist philosopher Zeno of Elea is due to Misra and Sudarshan [30], who were also the first to provide a consistent and rigorous mathematical framework. We can easily see that if the system is initially prepared in the unstable state $\left|\Phi_O\right\rangle$ and is monitored on its survival at the instants *0, t/n, 2t/n,….,(n-1)t/n, t* the probability for its survival is given by $P(t/n)^n$. In addition since the survival amplitude is differentiable and $\dot{P}(0)=0$, we can actually write for $n \to \infty$.

$$\left\{P(t/n)\right\}^n \cong \left\{P(0)+\dot{P}(0)(t/n)\right\}^n = 1 \tag{55}$$

independent of *t*. It is thus evident that the survival probability under discrete but frequent monitoring will be close to 1 provided that *t/n* is sufficiently small to observe departures from exponential decay law. The quantum Zeno effect is a direct consequence of general features of the Schrödinger equation that yield quadratic behavior of the survival probability at short times. Let *H* be the total Hamiltonian of a quantum system and Φ_O its initial state. The short time expansion of the survival probability yields a quadratic behavior of the form

$$P(t) \sim 1-t^2/\tau_z^2 \quad \text{where} \quad 1/\tau_z^2 \equiv \left\langle \Phi_O \right| H^2 \left| \Phi_O \right\rangle - \left\langle \Phi_O \right| H \left| \Phi_O \right\rangle^2 \tag{56}$$

and τ_z is the Zeno time, **[51]**. It is easy to see that $\left\{P(t/n)\right\}^n \cong \exp\left(-t^2/n\tau_z^2\right) \to 1 \ (as\ \mathrm{n} \to \infty)$. If the Hamiltonian is divided into the unperturbed H_O and the interaction part H_{int} , the Zeno time reads

$$1/\tau_z^2 \equiv \left\langle \Phi_O \right| {H_{\mathrm{int}}}^2 \left| \Phi_O \right\rangle = e\dot{s}_{t\ll}(0) \tag{57}$$

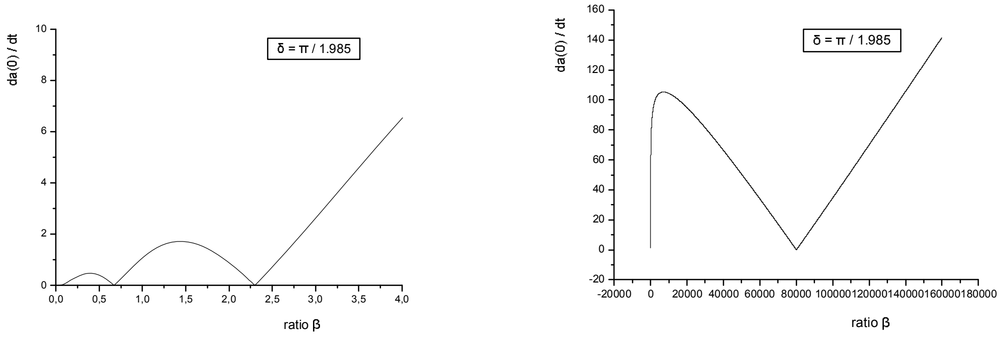
which in accordance to eq. (48) gives

$$1/\tau_z^2 \cong \frac{1}{2\tau_d^2}\left[\frac{\mathrm{sech}(\ln\beta)}{4} - N(\beta,\delta)\right] \mathrm{x} \left[\left(\tan^{-1}\beta-\delta\right)\left(1-\beta^2\right) - 2\beta\ln\frac{\cos\tan^{-1}\beta}{\cos\delta} + \tan\delta - \beta\right] \tag{58}$$

In this way we can estimate the Zeno time as

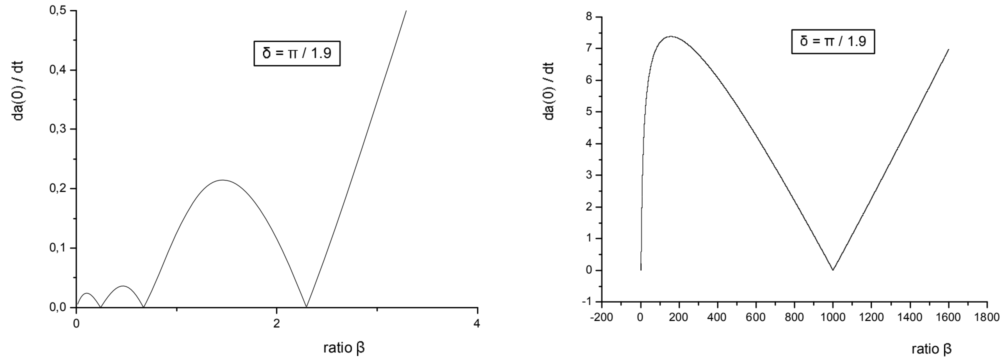
$$\tau_z \cong \frac{2\sqrt{2}\tau_d}{\sqrt{\mathrm{sech}(\ln\beta) - 4N(\beta,\delta)}}\left[\left(\tan^{-1}\beta-\delta\right)\left(1-\beta^2\right) - 2\beta\ln\frac{\cos\tan^{-1}\beta}{\cos\delta} + \tan\delta - \beta\right]^{-1/2} \tag{59}$$

Since the finiteness of the time derivative of the survival amplitude at *t=0* turns to be very important for the process of measurement, we may want to investigate its limits of accuracy as these are determined from eq. (52). Due to the structure of the *C(β)* coefficient $\dot{a}(0)$ is expected to have at least two roots, meaning the values of β_1 and β_2 that were previously revealed. This is important because these roots correspond to regions of quantity *β* where $\dot{a}(0)$ remains finite. We remind ourselves that these regions include resonant states where the non exponential decay dominates over the exponential, and according to the above quantum Zeno's paradox may appear. On the other hand eq. (52) describes the rate with which the measurement matrix element should change in the limit of very short times in order for the non exponential decay to be observed, as a function of the angle *δ(E)*. When the latter is close to π/2 then the complex energy shift does not significantly depend on the energy and the spectral distribution is almost Lorentzian line shape. For this reason we examine $\dot{a}(0)$ for different values of *δ(E)* close and distant from π/2, from π/1.985 to π/1.500. The figures that follow describe $\dot{a}(0)$ with *β*, for the previously mentioned values of the angle *δ*, covering each time two different regions of values of *β*.



(a) (b)

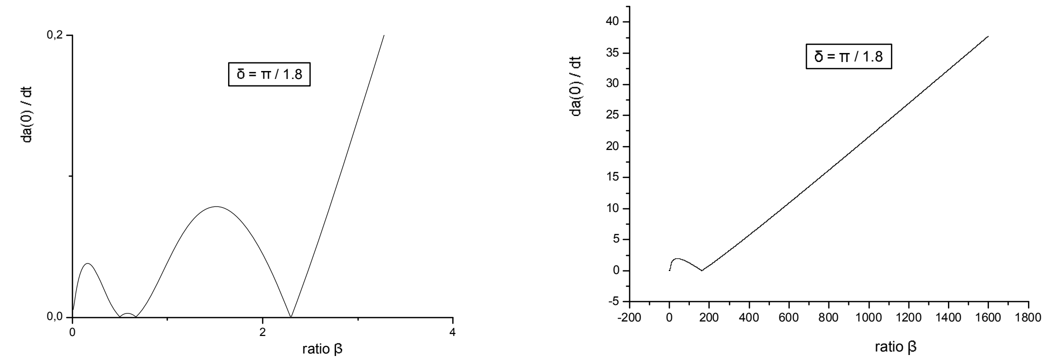
FIGURE 2. The variation of $\dot{a}(0)$ with *β* for an angle *δ = π/1.985*. a) For small values of *β* we have the two minima generated by *C(β)*, b) a third minimum appears for a value of *β* equal to 80x10^3. For greater values of *β*, $\dot{a}(0)$ approaches ∞.



(a) (b)

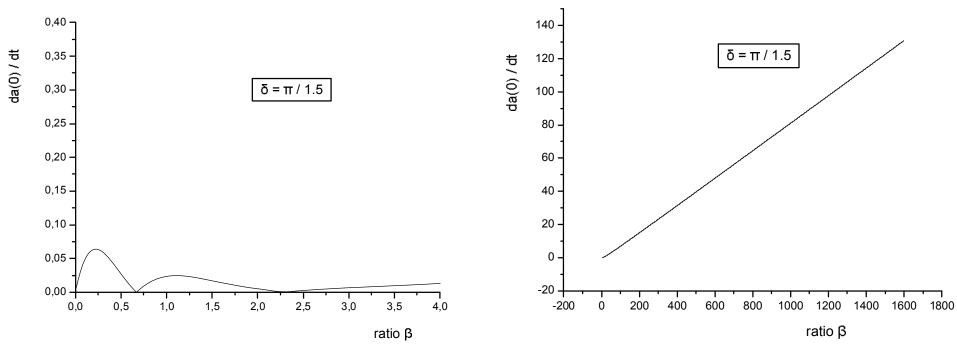
FIGURE 3. The variation of $\dot{a}(0)$ with *β* for an angle *δ = π/1.9*. a) For small values of *β* we have three minima, b) a fourth minimum appears for a value of *β* equal to 10^3. For greater values of *β*, $\dot{a}(0)$ approaches ∞.

The existence of minima in the above figures, ensures the finiteness of the time derivative of the survival amplitude, and enables the observation of the non exponential decay and generally the delay of propagation out of the initial state. We can see that when the angle *δ* is close to π/2 minima appear not only due to the structure of *C(β)* for small values of *β*, but for other values of *β* as well. This corresponds to the case where the Lorentzian line shape distribution provides a good approximation for the description of the unstable state. When *δ* further changes, there is no more space for other minima and $\dot{a}(0)$ approaches infinite as *β* grows. Thus the rate with which the measuring process varies tends to infinite even if the matrix element of the process is itself finite. In this way the variation of the measuring perturbation is discontinuous and we can not follow the system in this limit although the measurement might be in principle realizable.



(a) (b)

FIGURE 4. The variation of $\dot{a}(0)$ with *β* for an angle *δ* = *π/1.8*. a) For small values of *β* we have tho minima, since the third is getting absorbed b) a third minimum appears for a value of *β* equal to 180. For greater values of *β*, $\dot{a}(0)$ approaches ∞.



(a) (b)

FIGURE 5. The variation of $\dot{a}(0)$ with *β* for an angle *δ* = *π/1.5*. a) For small values of *β* we have again the two minima generated by *C(β)*, b) no other minimum appears for greater va-lues of *β* since the previous corresponding minimum is absorbed, and $\dot{a}(0)$ linearly approaches ∞.

When we treat the case of a strongly energy dependent energy shift, we can see from (46) that the rate with which the measuring perturbation changes, remains very small in magnitude for all the moments in time and also purely imaginary. The small magnitude has to do with the fact that in this case the non exponential decay domi-nates the propagation in the limits of very short times, due to the strong contribution of the angle *δ(E)*, [13,46], and so it is possible for the detector to follow the system and observe it. We have already seen that the *est* acquires a purely imaginary struc-ture for two reasons: $\mu(t)$ is the part of the Hamiltonian that causes the measuring perturbation of the system at *t=0* and in addition we can assume that the system can indeed be prepared in the state $|\Phi_o\rangle$ at that instant. Since the $est = \dot{a}(t)$ conserves this imaginary structure at later times also, it is clear that this is equivalent to say that we can repeatedly reset the system to its initial state. In this way we can in principle observe the non exponential decay and at the same time prevent decay by resetting the system back to Φ_O through repetitive observations, with a rate of change of the measuring process given by (46). Thinking so, (46) could serve as a definition for the frequency of observation needed, where for a wide class of systems, tests of nonde-cay repeated at arbitrarily small times would prevent the decay of an unstable state. Again this is not true for the complex spectral distribution for the same reasons explained in the previous paragraph.

Finally we may discuss the limit of very long times. As we have already mentioned the Paley-Wiener theorem [44], states that if the spectrum of the Hamiltonian is bounded below, so that $\rho(E) = 0$ for $E < 0$, then the survival amplitude decreases to 0 as $t \to \infty$ less rapidly than any exponential function. This is essentially Khalfin's argument [8] proving the necessity of deviation from the exponential decay law at large time. Looking at eq. (47) it is easy to understand that all stochastic terms containing $\exp\left(i\omega(\beta,\delta)t\right)$ produce cancellations and so the dominant term for *n=0* takes the following form for the two distributions

$$EST^{R,C}{}_{t>} \cong \frac{iN(\beta)C(\beta)\tau_d}{(1+i\beta)[(i\beta,2)-1]} \cdot \frac{1}{t^2} \tag{56}$$

Both terms rapidly go to zero as time grows, which makes it quite easy for the non exponential decay to be observed in this region of time. However and as we have already mentioned at the beginning of this section, a serious problem to be solved experimentally has mostly to do with the very small magnitude of the propagation at this region of time.

5. CONCLUDING REMARKS

In this work we discussed the origin and some of the properties of the deviations from exponential decay law, related to quantum resonances that can be described by Lorentzian line shape distributions. For this reason we first demonstrated the appearance of the Lorentzian distribution in both classical and quantum theory of resonances, as the mean amount of energy absorbed per unit time and the scattering phase respectively. In our study we distinguished two types of distributions both generated by the Lorentzian, namely the real truncated Lorentzian which is the semibounded Lorentzian, and the complex Lorentzian distribution which emerges when keeping only the appropriate *t>0* energy pole of the Lorentzian.

Following very general principles arising from kinematical effects, we showed that for large times and due to the fact that the decay products to whom the initial unstable state is finally distributed become uncorrelated, the kinematical factors dominate the distribution, giving a non exponential time dependence of the form $t^{-3/2}$. In the limit of very short times it is important to notice that the magnitude of the survival amplitude approaches the energy expectation value and the amplitude itself is purely imaginary. Then both the semiboundness of the spectrum and the finiteness of the expectation value of the energy, ensure that the time derivative of the survival probability approaches zero. Thus, at sufficient small time, the non decay probability falls off less rapidly than would be expected on the basis of the exponential decay law.

We constructed an analogous to the continuity equation in order to investigate the correlation between exponential and non exponential decay. We defined a spatial variable β , as twice the ratio of the energy position to the width function of the reso-nance and showed that there is an one by one correspondence between the strength of the exponential decay and this ratio. We can achieve the homogeneity of this equ-ation by applying a suitable measuring perturbation cancelling its second part, where the latter is defined as the exponential source term, *est*. The *est* expresses the corre-lation between exponential and non exponential decay and comes as the first time derivative of the previously mentioned measuring perturbation. For both types of distributions we found the same values of β close to 1 that zero the *est*, correspo-nding to the situation where the non exponential decay dominates the evolution. In the limit of short times we proved that the complex spectral distribution does not provide a complete theory for the observability of the non exponential decay, since it does not construct a purely imaginary survival amplitude as it should. Concerning the real distribution in the above limit, we produced the transcendental eq. (54) for the angle δ that describes the energy dependence of the complex energy shift as a function of the expectation value of the energy of the initial state. We also produced eq. (59) that gives the Zeno time as a function of the ratio β. We explored the finite-ness of the *est* term for other values of β as well, beyond those close to 1, since this permits someone to follow the system and observe the non exponential propagation, at least in principle. We showed that the finiteness is achieved only for specific valu-es of the angle δ close το π/2. For the case of a strongly energy dependent energy shift we found that the rate with which the measuring perturbation should change, is very small in magnitude for all the moments in time and also purely imaginary, conce-rning the real spectral distribution. The first characteristic was connected to the strong contribution of the non exponential decay while the second was connected to the potentiality to repeatedly reset the system to its initial state. In this way equation (46) provides information for the frequency needed for the repetition of the observa-tions. However this is not true for the complex spectral distribution, for the same reasons appeared in the short time regime.

In the limit of very long times both distributions produce *ests* whose dominant terms, meaning those beyond the stochastic terms that cancel each other, rapidly go to zero with time as $1/t^2$. This fact makes it quite easy for the non exponential decay to be observed. However the difficulty in this case has mostly to do with the very small magnitude of the propagation at this region of time.

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