Infinitesimal symmetry transformations of matrix-valued differential equations: An algebraic approach

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Abstract. The study of symmetries of partial differential equations (PDEs) has been traditionally treated as a geometrical problem. Although geometrical methods have been proven effective with regard to finding infinitesimal symmetry transformations, they present certain conceptual difficulties in the case of matrix-valued PDEs; for example, the usual differential-operator representation of the symmetry-generating vector fields is not possible in this case. An algebraic approach to the symmetry problem of PDEs is described, based on abstract operators (characteristic derivatives) which admit a standard differential-operator representation in the case of scalar-valued PDEs.

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1. Introduction

The problem of symmetries of a system of partial differential equations (PDEs) has been traditionally treated as a geometrical problem in the jet space of the independent and the dependent variables (including a sufficient number of partial derivatives of the latter variables with respect to the former ones). Two more or less equivalent approaches have been followed: (*a*) invariance of the system of PDEs itself, under infinitesimal transformations generated by corresponding vector fields in the jet space [1]; (*b*) invariance of a differential ideal of differential forms representing the system of PDEs, under the Lie derivative with respect to the vector fields representing the symmetry transformations [2-6].

Although effective with regard to calculating symmetries, these geometrical approaches suffer from a certain drawback of conceptual nature when it comes to matrix-valued PDEs. The problem is related to the inevitably mixed nature of the coordinates in the jet space (scalar independent variables versus matrix-valued dependent ones) and the need for a differential-operator representation of the symmetry vector fields. How does one define differentiation with respect to matrix-valued variables? Moreover, how does one calculate the Lie bracket of two differential operators in which some (or all) of the variables, as well as the coefficients of partial derivatives with respect to these variables, are matrices?

Although these difficulties were handled in some way in [4-6], it was eventually realized that an alternative, purely algebraic approach to the symmetry problem would be more appropriate in the case of matrix-valued PDEs. Elements of this approach were presented in [7] and later

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applied in particular problems [8-10]; no formal theoretical framework was fully developed, however.

An attempt to develop such a framework is made in this article. In Sections 2 and 3 we introduce the concept of the *characteristic derivative* – an abstract operator generalization of the Lie derivative used in scalar-valued problems – and we demonstrate the Lie-algebraic nature of the set of these derivatives.

The general symmetry problem for matrix-valued PDEs is presented in Sec. 4, and the Lie-algebraic property of symmetries of a PDE is proven in Sec. 5. In Sec. 6 we discuss the concept of a *recursion operator* [1,8-14] by which an infinite set of symmetries may in principle be produced from any known symmetry.

Finally, an application of these ideas is made in Sec. 7 by using the chiral field equation as an example.

To simplify our formalism, we restrict our analysis to the case of a single matrix-valued PDE in one dependent variable. Generalization to systems of PDEs is straightforward and is left to the reader (see, e.g., [1] for scalar-valued problems).

2. Total and characteristic derivative operators

A PDE for the unknown function $u=u(x^1, x^2, \dots) \equiv u(x^k)$ [where by (x^k) we collectively denote the independent variables $x^1, x^2, \dots$] is an expression of the form $F[u]=0$, where $F[u] \equiv F(x^k, u, u_k, u_{kl}, \dots)$ is a function in the *jet space* [1] of the independent variables (x^k), the dependent variable u, and the partial derivatives of various orders of u with respect to the x^k, which derivatives will be denoted by using subscripts: u_k, u_{kl}, u_{klm}, etc. A *solution* of the PDE is any function $u=\varphi(x^k)$ for which the relation $F[u]=0$ is satisfied at each point (x^k) in the domain of φ.

We assume that u, as well as all functions $F[u]$ in the jet space, are square-matrix-valued of fixed matrix dimensions. In particular, we require that, in its most general form, a function $F[u]$ in the jet space is expressible as a finite or an infinite sum of products of alternating x-dependent and u-dependent terms, of the form

$$F[u] = \sum a(x^k)\Pi[u]\,b(x^k)\Pi'[u]\,c(x^k)\cdots \tag{2.1}$$

where the $a(x^k)$, $b(x^k)$, $c(x^k)$, etc., are matrix-valued, and where the matrices $\Pi[u]$, $\Pi'[u]$, etc., are products of variables u, u_k, u_{kl}, etc., of the "fiber" space (or, more generally, products of powers of these variables). The set of all functions (2.1) is thus a (generally) non-commutative algebra.

If u is a scalar quantity, a total derivative operator can be defined in the usual way [1] as

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + u_{ijk} \frac{\partial}{\partial u_{jk}} + \cdots \tag{2.2}$$

where the summation convention over repeated up-and-down indices (such as j and k in this equation) has been adopted and will be used throughout. If, however, u is matrix-valued, the above expression is obviously not valid. A generalization of the definition of the total derivative is thus necessary for matrix-valued PDEs.

Definition 2.1. The *total derivative operator* with respect to the variable x^i is a linear operator D_i acting on functions $F[u]$ of the form (2.1) in the jet space and having the following properties:

1. On functions $f(x^k)$ in the base space, $D_i\, f(x^k) = \partial f / \partial x^i \equiv \partial_i\, f(x^k)$.

2. For $F[u]=u$, u_i, u_{ij}, etc., we have: $D_i\, u = u_i$, $D_i\, u_j = D_j\, u_i = u_{ij} = u_{ji}$, etc.

3. The operator D_i is a *derivation* on the algebra of all matrix-valued functions of the form (2.1) in the jet space; i.e., the *Leibniz rule* is satisfied:

$$D_i\left(F[u]G[u]\right) = \left(D_i F[u]\right)G[u] + F[u]D_i G[u] \tag{2.3}$$

Higher-order total derivatives $D_{ij}=D_iD_j$ may similarly be defined but they no longer possess the derivation property. Given that $\partial_i\partial_j=\partial_j\partial_i$ and that $u_{ij}=u_{ji}$, it follows that $D_iD_j = D_jD_i \Leftrightarrow D_{ij} = D_{ji}$; that is, total derivatives commute. We write: $[D_i , D_j]=0$, where, in general, $[A, B] \equiv AB–BA$ will denote the *commutator* of two operators or two matrices, as the case may be.

If u^{-1} is the inverse of u, such that $uu^{-1} = u^{-1}u = \mathbf{1}$, then we can define

$$D_i\left(u^{-1}\right) \equiv -u^{-1}\left(D_i u\right)u^{-1} \tag{2.4}$$

Moreover, for any functions $A[u]$ and $B[u]$ in the jet space, it can be shown that

$$D_i\,[A, B] = \left[D_i A, B\right] + \left[A, D_i B\right] \tag{2.5}$$

As an example, let $(x^1, x^2) \equiv (x, t)$ and let $F[u]=xtu_x^2$, where u is matrix-valued. Writing $F[u]=xtu_xu_x$, we have: $D_t\, F[u]=xu_x^2 + xt\,(u_{xt}\, u_x + u_x\, u_{xt})$.

Let now $Q[u] \equiv Q\,(x^k, u, u_k , u_{kl} , \ldots)$ be a function in the jet space. We will call this a *characteristic function* (or simply a *characteristic*) of a certain derivative, defined as follows:

Definition 2.2. The *characteristic derivative* with respect to $Q[u]$ is a linear operator Δ_Q acting on functions $F[u]$ in the jet space and having the following properties:

1. On functions $f(x^k)$ in the base space,

$$\Delta_Q\, f(x^k) = 0 \tag{2.6}$$

(that is, Δ_Q acts only in the fiber space).

2. For $F[u]=u$,

$$\Delta_Q u = Q[u] \tag{2.7}$$

3. Δ_Q commutes with total derivatives:

$$\Delta_Q D_i = D_i \Delta_Q \;\Leftrightarrow\; [\Delta_Q, D_i] = 0 \quad (\text{all } i) \tag{2.8}$$

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4. The operator Δ_Q is a *derivation* on the algebra of all matrix-valued functions of the form (2.1) in the jet space (the Leibniz rule is satisfied):

$$\Delta_Q\left(F[u]G[u]\right)=\left(\Delta_Q F[u]\right)G[u]+F[u]\Delta_Q G[u] \tag{2.9}$$

Corollary: By (2.7) and (2.8) we have:

$$\Delta_Q u_i=\Delta_Q D_i u=D_i Q[u] \tag{2.10}$$

We note that the operator Δ_Q is a well-defined quantity, in the sense that the action of Δ_Q on u uniquely determines the action of Δ_Q on any function $F[u]$ of the form (2.1) in the jet space. Moreover, since, by assumption, u and $Q[u]$ are matrices of equal dimensions, it follows from (2.7) that Δ_Q preserves the matrix character of u, as well as of any function $F[u]$ on which this operator acts.

We also remark that we have imposed conditions (2.6) and (2.8) having a certain property of symmetries of PDEs in mind; namely, *every* symmetry of a PDE can be represented by a transformation of the dependent variable u alone, i.e., can be expressed as a transformation in the fiber space (see [1], Chap. 5).

The following formulas, analogous to (2.4) and (2.5), may be written:

$$\Delta_Q\left(u^{-1}\right)\equiv -u^{-1}\left(\Delta_Q u\right)u^{-1} \tag{2.11}$$

$$\Delta_Q[A,B]=\left[\Delta_Q A,\,B\right]+\left[A,\,\Delta_Q B\right] \tag{2.12}$$

As an example, let $(x^1, x^2)\equiv(x, t)$ and let $F[u]=a(x,t)u^2b(x,t)+[u_x , u_t]$, where a, b and u are matrices of equal dimensions. Writing $u^2=uu$ and using properties (2.7), (2.9), (2.10) and (2.12), we find: $\Delta_Q F[u]=a(x,t)(Qu+uQ)b(x,t)+[D_x Q,\, u_t]+[u_x ,\, D_t Q]$.

In the case where u is scalar-valued (thus so is $Q[u]$), the characteristic derivative Δ_Q admits a differential-operator representation of the form

$$\Delta_Q=Q[u]\frac{\partial}{\partial u}+\left(D_i Q[u]\right)\frac{\partial}{\partial u_i}+\left(D_i D_j Q[u]\right)\frac{\partial}{\partial u_{ij}}+\cdots \tag{2.13}$$

(See [1], Chap. 5, for an analytic proof of property (2.8) in this case.)

3. The Lie algebra of characteristic derivatives

The characteristic derivatives Δ_Q acting on functions $F[u]$ of the form (2.1) in the jet space constitute a *Lie algebra of derivations* on the algebra of the $F[u]$. The proof of this statement is contained in the following three Propositions.

Proposition 3.1. Let Δ_Q be a characteristic derivative with respect to the characteristic $Q[u]$; i.e., $\Delta_Q u=Q[u]$ [cf. Eq. (2.7)]. Let λ be a constant (real or complex). We define the operator $\lambda\Delta_Q$ by the relation

$$(\lambda\Delta_Q)F[u] \equiv \lambda\,(\Delta_Q F[u])\,.$$

Then, $\lambda\Delta_Q$ is a characteristic derivative with characteristic $\lambda Q[u]$. That is,

$$\lambda\,\Delta_Q = \Delta_{\lambda Q} \tag{3.1}$$

Proof. (*a*) The operator $\lambda\Delta_Q$ is linear, since so is Δ_Q .

(*b*) For $F[u]=u$, $(\lambda\Delta_Q)u=\lambda(\Delta_Q u)=\lambda Q[u]$.

(*c*) $\lambda\Delta_Q$ commutes with total derivatives D_i , since so does Δ_Q .

(*d*) Given that Δ_Q satisfies the Leibniz rule (2.9), it is easily shown that so does $\lambda\Delta_Q$.

Comment: Condition (*c*) would not be satisfied if we allowed λ to be a function of the x^k, instead of being a constant, since $\lambda(x^k)$ generally does not commute with the D_i. Therefore, relation (3.1) is not valid for a non-constant λ.

Proposition 3.2. Let Δ_1 and Δ_2 be characteristic derivatives with respect to the characteristics $Q_1[u]$ and $Q_2[u]$, respectively; i.e., $\Delta_1 u=Q_1[u]$, $\Delta_2 u=Q_2[u]$. We define the operator $\Delta_1+\Delta_2$ by

$$(\Delta_1+\Delta_2)\,F[u] \equiv \Delta_1\,F[u] + \Delta_2\,F[u]\,.$$

Then, $\Delta_1+\Delta_2$ is a characteristic derivative with characteristic $Q_1[u]+Q_2[u]$. That is,

$$\Delta_1 + \Delta_2 = \Delta_Q \quad \text{with} \quad Q[u] = Q_1[u] + Q_2[u] \tag{3.2}$$

Proof. (*a*) The operator $\Delta_1+\Delta_2$ is linear, as a sum of linear operators.

(*b*) For $F[u]=u$, $(\Delta_1+\Delta_2)u = \Delta_1 u+\Delta_2 u = Q_1[u]+Q_2[u]$.

(*c*) $\Delta_1+\Delta_2$ commutes with total derivatives D_i , since so do Δ_1 and Δ_2 .

(*d*) Given that each of Δ_1 and Δ_2 satisfies the Leibniz rule (2.9), it is not hard to show that the same is true for $\Delta_1+\Delta_2$.

Proposition 3.3. Let Δ_1 and Δ_2 be characteristic derivatives with respect to the characteristics $Q_1[u]$ and $Q_2[u]$, respectively; i.e., $\Delta_1 u=Q_1[u]$, $\Delta_2 u=Q_2[u]$. We define the operator $[\Delta_1 , \Delta_2]$ (*Lie bracket* of Δ_1 and Δ_2) by

$$[\Delta_1\,,\Delta_2]\,F[u] \equiv \Delta_1\,(\Delta_2\,F[u]) - \Delta_2\,(\Delta_1\,F[u])\,.$$

Then, $[\Delta_1 , \Delta_2]$ is a characteristic derivative with characteristic $\Delta_1 Q_2[u]-\Delta_2 Q_1[u]$. That is,

$$[\Delta_1, \Delta_2] = \Delta_Q \quad \text{with} \quad Q[u] = \Delta_1 Q_2[u] - \Delta_2 Q_1[u] \equiv Q_{1,2}[u] \tag{3.3}$$

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Proof. (*a*) The linearity of $[\Delta_1, \Delta_2]$ follows from the linearity of Δ_1 and Δ_2.

(*b*) For $F[u]=u$, $[\Delta_1, \Delta_2]u = \Delta_1(\Delta_2 u) - \Delta_2(\Delta_1 u) = \Delta_1 Q_2[u] - \Delta_2 Q_1[u] \equiv Q_{1,2}[u]$.

(*c*) $[\Delta_1, \Delta_2]$ commutes with total derivatives D_i, since so do Δ_1 and Δ_2.

(*d*) Given that each of Δ_1 and Δ_2 satisfies the Leibniz rule (2.9), one can show (after some algebra and cancellation of terms) that the same is true for $[\Delta_1, \Delta_2]$.

In the case where u (thus the Q's also) is scalar-valued, the Lie bracket admits a standard differential-operator representation [1]:

$$[\Delta_1, \Delta_2] = Q_{1,2}[u]\frac{\partial}{\partial u} + \left(D_i Q_{1,2}\right)\frac{\partial}{\partial u_i} + \left(D_i D_j Q_{1,2}\right)\frac{\partial}{\partial u_{ij}} + \cdots \qquad (3.4)$$

where $Q_{1,2}[u] = [\Delta_1, \Delta_2]u = \Delta_1 Q_2[u] - \Delta_2 Q_1[u]$.

The Lie bracket $[\Delta_1, \Delta_2]$ has the following properties:

1. $[\Delta_1, a\Delta_2 + b\Delta_3] = a[\Delta_1, \Delta_2] + b[\Delta_1, \Delta_3]$;

   $[a\Delta_1 + b\Delta_2, \Delta_3] = a[\Delta_1, \Delta_3] + b[\Delta_2, \Delta_3]$ $\quad$ (a, b = const.)

2. $[\Delta_1, \Delta_2] = -[\Delta_2, \Delta_1]$ $\quad$ (*antisymmetry*)

3. $[\Delta_1, [\Delta_2, \Delta_3]] + [\Delta_2, [\Delta_3, \Delta_1]] + [\Delta_3, [\Delta_1, \Delta_2]] = 0$;

   $[[\Delta_1, \Delta_2], \Delta_3] + [[\Delta_2, \Delta_3], \Delta_1] + [[\Delta_3, \Delta_1], \Delta_2] = 0$ $\quad$ (*Jacobi identity*)

4. The symmetry problem for PDEs

Let $F[u]=0$ be a PDE in the independent variables $x^k \equiv x^1, x^2, \ldots$, and the (generally) matrix-valued dependent variable u. A transformation $u(x^k) \to u'(x^k)$ from the function u to a new function u' represents a *symmetry* of the PDE if the following condition is satisfied: $u'(x^k)$ is a solution of $F[u]=0$ *when* $u(x^k)$ is a solution; that is, $F[u']=0$ when $F[u]=0$.

We will restrict our attention to *continuous symmetries*, which can be expressed as infinitesimal transformations. Although such symmetries may involve transformations of the independent variables (x^k), they may equivalently be expressed as transformations of u alone (see [1], Chap. 5), i.e., as transformations in the fiber space.

An infinitesimal symmetry transformation is written symbolically as

$$u \to u' = u + \delta u$$

where δu is an infinitesimal quantity, in the sense that all powers $(\delta u)^n$ with $n>1$ may be neglected. The *symmetry condition* is thus written

$$F[u+\delta u] = 0 \quad \text{when} \quad F[u] = 0 \qquad (4.1)$$

An infinitesimal change δu of u induces a change $\delta F[u]$ of $F[u]$, where

$$\delta F[u] = F[u+\delta u] - F[u] \quad \Leftrightarrow \quad F[u+\delta u] = F[u] + \delta F[u] \tag{4.2}$$

Now, if δu is an infinitesimal symmetry and if u is a solution of $F[u]=0$, then $u+\delta u$ also is a solution; that is, $F[u+\delta u]=0$. This means that $\delta F[u]=0$ when $F[u]=0$. The symmetry condition (4.1) is thus written as follows:

$$\delta F[u]=0 \mod F[u] \tag{4.3}$$

A symmetry transformation (we denote it M) of the PDE $F[u]=0$ produces a one-parameter family of solutions of the PDE from any given solution $u(x^k)$. We express this by writing

$$M: u(x^k) \to \bar{u}(x^k;\alpha) \quad \text{with} \quad \bar{u}(x^k;0)=u(x^k) \tag{4.4}$$

For infinitesimal values of the parameter α,

$$\bar{u}(x^k;\alpha) \simeq u(x^k) + \alpha Q[u] \quad \text{where} \quad Q[u] = \left.\frac{d\bar{u}}{d\alpha}\right|_{\alpha=0} \tag{4.5}$$

The function $Q[u] \equiv Q(x^k, u, u_k, u_{kl}, \ldots)$ in the jet space is called the *characteristic* of the symmetry (or, the *symmetry characteristic*). Putting

$$\delta u = \bar{u}(x^k;\alpha) - u(x^k) \tag{4.6}$$

we write, for infinitesimal α,

$$\delta u = \alpha Q[u] \tag{4.7}$$

We notice that the infinitesimal operator δ has the following properties:

1. According to its definition (4.2), δ is a linear operator:

$$\delta(F[u]+G[u]) = (F[u+\delta u]+ G[u+\delta u]) - (F[u]+G[u]) = \delta F[u]+\delta G[u] \,.$$

2. By assumption regarding the nature of our symmetry transformations, δ produces changes in the fiber space while it doesn't affect functions $f(x^k)$ in the base space [this is implicitly stated in (4.6)].

3. Since δ represents a difference, it commutes with all total derivatives D_i:

$$\delta(D_i A[u]) = D_i(\delta A[u]) \,.$$

In particular, for $A[u]=u$,

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$$\delta u_i = \delta(D_i u) = D_i(\delta u) = \alpha D_i Q[u] ,$$

where we have used (4.7).

4. Since δ expresses an infinitesimal change, it may be approximated to a differential; in particular, it satisfies the Leibniz rule:

$$\delta(A[u]B[u]) = (\delta A[u])B[u] + A[u]\delta B[u] .$$

For example, $\delta(u^2) = \delta(uu) = (\delta u)u + u\delta u = \alpha(Qu + uQ)$.

Now, consider the characteristic derivative Δ_Q with respect to the symmetry characteristic $Q[u]$. According to (2.7),

$$\Delta_Q u = Q[u] \tag{4.8}$$

We observe that the infinitesimal operator δ and the characteristic derivative Δ_Q share common properties. From (4.7) and (4.8) it follows that the two linear operators are related by

$$\delta u = \alpha \Delta_Q u \tag{4.9}$$

and, by extension,

$$\delta u_i = \alpha D_i Q[u] = \alpha \Delta_Q u_i \text{ , etc.}$$

[see (2.10)]. Moreover, for scalar-valued u and by the infinitesimal character of the operator δ, we may write:

$$\delta F[u] = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_i}\delta u_i + \cdots = \alpha\left(\frac{\partial F}{\partial u}Q[u] + \frac{\partial F}{\partial u_i}D_i Q[u] + \frac{\partial F}{\partial u_{ij}}D_i D_j Q[u] + \cdots\right)$$

while, by (2.13),

$$\Delta_Q F[u] = \frac{\partial F}{\partial u}Q[u] + \frac{\partial F}{\partial u_i}D_i Q[u] + \frac{\partial F}{\partial u_{ij}}D_i D_j Q[u] + \cdots \tag{4.10}$$

The above observations lead us to the conclusion that, in general, the following relation is true:

$$\delta F[u] = \alpha \Delta_Q F[u] \tag{4.11}$$

The symmetry condition (4.3) is thus written:

$$\Delta_Q F[u] = 0 \mod F[u] \tag{4.12}$$

In particular, if u is scalar-valued, the above condition is written

$$\frac{\partial F}{\partial u}Q[u]+\frac{\partial F}{\partial u_i}D_i Q[u]+\frac{\partial F}{\partial u_{ij}}D_i D_j Q[u]+\cdots=0 \quad \text{mod} \quad F[u] \tag{4.13}$$

which is a linear PDE for $Q[u]$. More generally, for matrix-valued u and for a function $F[u]$ of the form (2.1), the symmetry condition for the PDE $F[u]=0$ is a *linear* PDE for the symmetry characteristic $Q[u]$. We write this PDE symbolically as

$$S(Q\,;u) \equiv \Delta_Q F[u]=0 \quad \text{mod} \quad F[u] \tag{4.14}$$

where the function $S(Q\,;u)$ is linear in Q and all total derivatives of Q. (The linearity of S in Q follows from the Leibniz rule and the specific form (2.1) of $F[u]$.)

Below is a list of formulas that may be useful in calculations:

- $\Delta_Q u_i = D_i Q[u]\,, \quad \Delta_Q u_{ij} = D_i D_j Q[u]\,, \quad \text{etc.}$
- $\Delta_Q u^2 = \Delta_Q (uu) = Q[u]u+uQ[u] \quad \text{(etc.)}$
- $\Delta_Q (u^{-1}) = -\,u^{-1} (\Delta_Q u)\, u^{-1} = -\,u^{-1}\, Q[u]\, u^{-1}$
- $\Delta_Q [A[u]\,, B[u]] = [\Delta_Q A\,, B] + [A\,, \Delta_Q B]$

Comment: According to (4.12), $\Delta_Q F[u]$ vanishes if $F[u]$ vanishes. Given that Δ_Q is a linear operator, the reader may wonder whether this condition is identically satisfied (a linear operator acting on a zero function always produces a zero function!). Note, however, that the function $F[u]$ is *not identically* zero; it becomes zero only *for solutions* of the given PDE. What we need to do, therefore, is to first evaluate $\Delta_Q F[u]$ for *arbitrary* u and *then* demand that the result vanish *when* u is a solution of the PDE $F[u]=0$.

An alternative – and perhaps more transparent – version of the symmetry condition (4.12) is the requirement that the following relation be satisfied:

$$\Delta_Q F[u]=\hat{L}\,F[u] \tag{4.15}$$

where $\hat{L}$ is a linear operator acting on functions in the jet space (see, e.g., [1], Chap. 2 and 5, for a rigorous justification of this condition in the case of scalar-valued PDEs). For example, one may have

$$\Delta_Q F[u]=\sum_i \beta_i (x^k) D_i F[u]+\sum_{i,j}\gamma_{ij}(x^k) D_i D_j F[u]+A(x^k)F[u]+F[u]B(x^k)$$

where the β_i and γ_{ij} are scalar-valued, while A and B are matrix-valued. Let us see some examples, restricting for the moment our attention to scalar PDEs.

Example 4.1. The *sine-Gordon (s-G) equation* is written

$$F[u] \equiv u_{xt} - \sin u = 0\,.$$

Here, $(x^1, x^2) \equiv (x, t)$. Since $\sin u$ can be expanded into an infinite series in powers of u, we see that $F[u]$ has the required form (2.1). Moreover, since u is a scalar function, we can write the symmetry condition by using (4.13):

$$S(Q;u) \equiv Q_{xt} - (\cos u)\, Q = 0 \quad \text{mod} \quad F[u]$$

where $S(Q; u)= \Delta_Q F[u]$ and where by subscripts we denote total differentiations with respect to the indicated variables. Let us verify the solution $Q[u]=u_x$. This characteristic corresponds to the symmetry transformation [cf. Eq. (4.4)]

$$M:\ u(x,t) \rightarrow \bar{u}(x,t;\alpha) = u(x+\alpha,\, t) \tag{4.16}$$

which implies that, if $u(x,t)$ is a solution of the s-G equation, then $\bar{u}(x,t) = u(x+\alpha,\, t)$ also is a solution. We have:

$$Q_{xt} - (\cos u)\, Q = (u_x)_{xt} - (\cos u)\, u_x = (u_{xt} - \sin u)_x = D_x F[u] = 0 \quad \text{mod} \quad F[u]\ .$$

Notice that $\Delta_Q F[u]$ is of the form (4.15), with $\hat{L} \equiv D_x$. Similarly, the characteristic $Q[u] = u_t$ corresponds to the symmetry

$$M:\ u(x,t) \rightarrow \bar{u}(x,t;\alpha) = u(x,\, t+\alpha) \tag{4.17}$$

That is, if $u(x,t)$ is a solution of the s-G equation, then so is $\bar{u}(x,t) = u(x,\, t+\alpha)$. The symmetries (4.16) and (4.17) reflect the fact that the s-G equation does not contain the variables x and t explicitly. (Of course, this equation has many more symmetries which are not displayed here; see, e.g., [1].)

Example 4.2. The *heat equation* is written

$$F[u] \equiv u_t - u_{xx} = 0\ .$$

The symmetry condition (4.13) reads

$$S(Q;u) \equiv Q_t - Q_{xx} = 0 \quad \text{mod} \quad F[u]$$

where $S(Q; u)= \Delta_Q F[u]$. As is easy to show, the symmetries (4.16) and (4.17) are valid here, too. Let us now try the solution $Q[u]=u$. We have:

$$Q_t - Q_{xx} = u_t - u_{xx} = F[u] = 0 \quad \text{mod} \quad F[u]\ .$$

This symmetry corresponds to the transformation

$$M:\ u(x,t) \rightarrow \bar{u}(x,t;\alpha) = e^{\alpha}\, u(x,t) \tag{4.18}$$

and is a consequence of the linearity of the heat equation.

Example 4.3. One form of the *Burgers equation* is

$$F[u] \equiv u_t - u_{xx} - u_x^{\,2} = 0 \, .$$

The symmetry condition (4.13) is written

$$S(Q;u) \equiv Q_t - Q_{xx} - 2u_x Q_x = 0 \mod F[u]$$

where, as always, $S(Q; u)=\Delta_Q F[u]$. Putting $Q= u_x$ and $Q= u_t$, we verify the symmetries (4.16) and (4.17):

$$Q_t - Q_{xx} - 2u_x Q_x = u_{xt} - u_{xxx} - 2u_x u_{xx} = D_x F[u] = 0 \mod F[u]$$
$$Q_t - Q_{xx} - 2u_x Q_x = u_{tt} - u_{xxt} - 2u_x u_{xt} = D_t F[u] = 0 \mod F[u]$$

Note again that $\Delta_Q F[u]$ is of the form (4.15), with $\hat{L} \equiv D_x$ and $\hat{L} \equiv D_t$. Another symmetry is $Q[u]=1$, which corresponds to the transformation

$$M: \; u(x,t) \rightarrow \bar{u}(x,t;\alpha) = u(x,t) + \alpha \tag{4.19}$$

and is a consequence of the fact that u enters $F[u]$ only through its derivatives.

Example 4.4. The *wave equation* is written

$$F[u] \equiv u_{tt} - c^2 u_{xx} = 0 \quad (c = \text{const.})$$

and its symmetry condition reads

$$S(Q;u) \equiv Q_{tt} - c^2 Q_{xx} = 0 \mod F[u] \, .$$

The solution $Q[u] = x\,u_x + t\,u_t$ corresponds to the symmetry transformation

$$M: \; u(x,t) \rightarrow \bar{u}(x,t;\alpha) = u(e^{\alpha}x, e^{\alpha}t) \tag{4.20}$$

expressing the invariance of the wave equation under a scale change of x and t. [The reader may show that the transformations (4.16) – (4.19) also express symmetries of the wave equation.]

It is remarkable that each of the above PDEs admits an infinite set of symmetry transformations [1]. An effective method for finding such infinite sets is the use of a *recursion operator*, which produces a new symmetry characteristic every time it acts on a known characteristic. More will be said on recursion operators in Sec. 6.

5. The Lie algebra of symmetries

As is well known [1], the set of symmetries of a PDE $F[u]=0$ has the structure of a Lie algebra. Let us demonstrate this property in the context of our abstract formalism.

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Proposition 5.1. Let $\mathcal{L}$ be the set of characteristic derivatives Δ_Q with respect to the symmetry characteristics $Q[u]$ of the PDE $F[u]=0$. The set $\mathcal{L}$ is a (finite- or infinite-dimensional) Lie subalgebra of the Lie algebra of characteristic derivatives acting on functions $F[u]$ in the jet space (cf. Sec. 3).

Proof. (*a*) Let $\Delta_Q \in \mathcal{L} \Leftrightarrow \Delta_Q F[u]=0$ (mod $F[u]$). If λ is a constant (real or complex, depending on the nature of the problem) then $(\lambda\Delta_Q)F[u] \equiv \lambda\Delta_Q F[u]=0$, which means that $\lambda\Delta_Q \in \mathcal{L}$. Given that $\lambda\Delta_Q = \Delta_{\lambda Q}$ [see Eq. (3.1)] we conclude that, if $Q[u]$ is a symmetry characteristic of $F[u]=0$, then so is $\lambda Q[u]$.

(*b*) Let $\Delta_1 \in \mathcal{L}$ and $\Delta_2 \in \mathcal{L}$ be characteristic derivatives with respect to the symmetry characteristics $Q_1[u]$ and $Q_2[u]$, respectively. Then, $\Delta_1 F[u]=0$, $\Delta_2 F[u]=0$, and hence, $(\Delta_1+\Delta_2)F[u] \equiv \Delta_1 F[u]+\Delta_2 F[u]=0$; therefore, $(\Delta_1+\Delta_2)\in\mathcal{L}$. It also follows from Eq. (3.2) that, if $Q_1[u]$ and $Q_2[u]$ are symmetry characteristics of $F[u]=0$, then so is their sum $Q_1[u]+Q_2[u]$.

(*c*) Let $\Delta_1 \in \mathcal{L}$ and $\Delta_2 \in \mathcal{L}$, as above. Then, by (4.15),

$$\Delta_1 F[u] = \hat{L}_1 F[u], \quad \Delta_2 F[u] = \hat{L}_2 F[u].$$

Now, by the definition of the Lie bracket and the linearity of both Δ_i and $\hat{L}_i$ $(i=1,2)$ we have:

$$[\Delta_1, \Delta_2]F[u] = \Delta_1(\Delta_2 F[u]) - \Delta_2(\Delta_1 F[u]) = \Delta_1(\hat{L}_2 F[u]) - \Delta_2(\hat{L}_1 F[u])$$
$$\equiv (\Delta_1 \hat{L}_2 - \Delta_2 \hat{L}_1)F[u] = 0 \mod F[u]$$

We thus conclude that $[\Delta_1, \Delta_2]\in\mathcal{L}$. Moreover, it follows from Eq. (3.3) that, if $Q_1[u]$ and $Q_2[u]$ are symmetry characteristics of $F[u]=0$, then so is the function

$$Q_{1,2}[u] = \Delta_1 Q_2[u] - \Delta_2 Q_1[u] .$$

Assume now that the PDE $F[u]=0$ has an n-dimensional symmetry algebra $\mathcal{L}$ (which may be a finite subalgebra of an infinite-dimensional symmetry Lie algebra). Let $\{\Delta_1, \Delta_2, \ldots, \Delta_n\} \equiv \{\Delta_k\}$, with corresponding symmetry characteristics $\{Q_k\}$, be a set of n linearly independent operators that constitute a basis of $\mathcal{L}$, and let Δ_i , Δ_j be any two elements of this basis. Given that $[\Delta_i, \Delta_j]\in\mathcal{L}$, this Lie bracket must be expressible as a linear combination of the $\{\Delta_k\}$, with constant coefficients. We write

$$[\Delta_i, \Delta_j] = \sum_{k=1}^{n} c_{ij}^k \Delta_k \tag{5.1}$$

where the coefficients of the Δ_k in the sum are the antisymmetric *structure constants* of the Lie algebra $\mathcal{L}$ in the basis $\{\Delta_k\}$.

The operator relation (5.1) can be expressed in an equivalent, characteristic form by allowing the operators on both sides to act on u and by using the fact that $\Delta_k u=Q_k[u]$:

$$[\Delta_i, \Delta_j]u = \left(\sum_{k=1}^{n} c_{ij}^k \Delta_k\right)u = \sum_{k=1}^{n} c_{ij}^k (\Delta_k u) \Rightarrow$$

$$\Delta_i Q_j[u] - \Delta_j Q_i[u] = \sum_{k=1}^{n} c_{ij}^k Q_k[u] \tag{5.2}$$

Example 5.1. One of the several forms of the *Korteweg-de Vries (KdV) equation* is

$$F[u] \equiv u_t + uu_x + u_{xxx} = 0 .$$

The symmetry condition (4.14) is written

$$S(Q;u) \equiv Q_t + Q\,u_x + u\,Q_x + Q_{xxx} = 0 \quad \text{mod} \quad F[u] \tag{5.3}$$

where $S(Q;u)=\Delta_Q F[u]$. The KdV equation admits a symmetry Lie algebra of infinite dimensions [1]. This algebra has a finite, 4-dimensional subalgebra $\mathcal{L}$ of *point transformations*. A symmetry operator (characteristic derivative) Δ_Q is determined by its corresponding characteristic $Q[u]=\Delta_Q u$. Thus, a basis $\{\Delta_1 ,..., \Delta_4\}$ of $\mathcal{L}$ corresponds to a set of four independent characteristics $\{Q_1 ,..., Q_4\}$. Such a basis of characteristics is the following:

$$Q_1[u]= u_x , \quad Q_2[u]= u_t , \quad Q_3[u]= tu_x - 1 , \quad Q_4[u]= xu_x + 3tu_t + 2u$$

The $Q_1 ,..., Q_4$ satisfy the PDE (5.3), since, as we can show,

$$S(Q_1;u) = D_x F[u] , \quad S(Q_2;u) = D_t F[u] , \quad S(Q_3;u) = tD_x F[u] ,$$
$$S(Q_4;u) = (5 + xD_x + 3tD_t) F[u]$$

[Note once more that $\Delta_Q F[u]$ is of the form (4.15) in each case.] Let us now see two examples of calculating the structure constants of $\mathcal{L}$ by application of (5.2). We have:

$$\Delta_1 Q_2 - \Delta_2 Q_1 = \Delta_1 u_t - \Delta_2 u_x = (\Delta_1 u)_t - (\Delta_2 u)_x = (Q_1)_t - (Q_2)_x = (u_x)_t - (u_t)_x = 0$$
$$\equiv \sum_{k=1}^{4} c_{12}^k Q_k$$

Since the Q_k are linearly independent, we must necessarily have $c_{12}^k = 0$, $k = 1,2,3,4$. Also,

$$\Delta_2 Q_3 - \Delta_3 Q_2 = \Delta_2 (tu_x - 1) - \Delta_3 u_t = t(\Delta_2 u)_x - (\Delta_3 u)_t = t(Q_2)_x - (Q_3)_t$$
$$= tu_{tx} - (u_x + tu_{xt}) = -u_x = -Q_1 \equiv \sum_{k=1}^{4} c_{23}^k Q_k$$

Therefore, $c_{23}^{1}=-1$, $c_{23}^{2}=c_{23}^{3}=c_{23}^{4}=0$.

6. Recursion operators

Let $\delta u=\alpha Q[u]$ be an infinitesimal symmetry of the PDE $F[u]=0$, where $Q[u]$ is the symmetry characteristic. For any solution $u(x^k)$ of this PDE, the function $Q[u]$ satisfies the linear PDE

$$S(Q;u)\equiv \Delta_Q F[u]=0 \tag{6.1}$$

Because of the linearity of (6.1) in Q, the sum $Q_1[u]+Q_2[u]$ of two solutions of this PDE, as well as the multiple $\lambda Q[u]$ of any solution by a constant, also are solutions of (6.1) for a given u. Thus, for any solution u of $F[u]=0$, the solutions $\{Q[u]\}$ of the PDE (6.1) form a linear space, which we call S_u.

A *recursion operator* $\hat{R}$ is a linear operator that maps the space S_u into itself. Thus, if $Q[u]$ is a symmetry characteristic of $F[u]=0$ (i.e., a solution of (6.1) for a given u) then so is $\hat{R}Q[u]$:

$$S(\hat{R}Q;u)=0 \quad \text{when} \quad S(Q;u)=0 \tag{6.2}$$

Obviously, any power of a recursion operator also is a recursion operator. Thus, starting with any symmetry characteristic $Q[u]$, one may in principle obtain an infinite set of such characteristics by repeated application of the recursion operator.

A new approach to recursion operators was suggested in the early 1990s [11,15-17] (see also [8-10]) according to which a recursion operator may be viewed as an *auto-Bäcklund transformation* (BT) [18] for the symmetry condition (6.1) of the PDE $F[u]=0$. By integrating the BT, one obtains new solutions $Q'[u]$ of the linear PDE (6.1) from known ones, $Q[u]$. Typically, this type of recursion operator produces *nonlocal* symmetries in which the symmetry characteristic depends on *integrals* (rather than derivatives) of u. This approach proved to be particularly effective for matrix-valued PDEs such as the nonlinear self-dual Yang-Mills equation, of which new infinite-dimensional sets of “potential symmetries” were discovered [9,11,15].

7. An example: The chiral field equation

Let us consider the *chiral field equation*

$$F[g]\equiv (g^{-1}g_x)_x+(g^{-1}g_t)_t=0 \tag{7.1}$$

where, in general, subscripts x and t denote total derivatives D_x and D_t, respectively, and where g is a $GL(n,C)$-valued function of x and t, i.e., a complex, non-singular $(n\times n)$ matrix function, differentiable for all x and t. Let $\delta g=\alpha Q[g]$ be an infinitesimal symmetry transformation for the PDE (7.1), with symmetry characteristic $Q[g]=\Delta_Q g$. It is convenient to put

$$Q[g] = g\,\Phi[g] \quad \Leftrightarrow \quad \Phi[g] = g^{-1}Q[g]\ .$$

The symmetry condition for (7.1) is

$$\Delta_Q F[g] = 0 \quad \text{mod} \quad F[g]\ .$$

This condition will yield a linear PDE for Q or, equivalently, a linear PDE for Φ. By using the properties of the characteristic derivative, we find the latter PDE to be

$$S(\Phi; g) \equiv D_x\left(\Phi_x + [g^{-1}g_x, \Phi]\right) + D_t\left(\Phi_t + [g^{-1}g_t, \Phi]\right) = 0 \ \text{ mod } \ F[g] \qquad (7.2)$$

where, as usual, square brackets denote commutators of matrices.

A useful identity that will be needed in the sequel is the following:

$$(g^{-1}g_t)_x - (g^{-1}g_x)_t + [g^{-1}g_x,\, g^{-1}g_t] = 0 \qquad (7.3)$$

Let us first consider symmetry transformations in the base space, i.e., coordinate transformations of x, t. An obvious symmetry is x-translation, $x'=x+\alpha$, given that the PDE (7.1) does not contain the independent variable x explicitly. For infinitesimal values of the parameter α, we write $\delta x=\alpha$. The symmetry characteristic is $Q[g]=g_x$, so that $\Phi[g]=g^{-1}g_x$. By substituting this expression for Φ into the symmetry condition (7.2) and by using the identity (7.3), we can verify that (7.2) is indeed satisfied:

$$S(\Phi; g) = D_x F[g] = 0 \quad \text{mod} \quad F[g]\ .$$

Similarly, for t-translation, $t'=t+\alpha$ (infinitesimally, $\delta t=\alpha$) with $Q[g]=g_t$, we find

$$S(\Phi; g) = D_t F[g] = 0 \quad \text{mod} \quad F[g]\ .$$

Another obvious symmetry of (7.1) is a scale change of both x and t: $x'=\lambda x$, $t'=\lambda t$. Setting $\lambda=1+\alpha$, where α is infinitesimal, we write $\delta x=\alpha x$, $\delta t=\alpha t$. The symmetry characteristic is $Q[g]=xg_x+tg_t$, so that $\Phi[g]=xg^{-1}g_x+tg^{-1}g_t$. Substituting for Φ into the symmetry condition (7.2) and using the identity (7.3) where necessary, we find that

$$S(\Phi; g) = \left(2 + xD_x + tD_t\right) F[g] = 0 \quad \text{mod} \quad F[g]\ .$$

Let us call $Q_1[g]=g_x$, $Q_2[g]=g_t$, $Q_3[g]=xg_x+tg_t$, and let us consider the corresponding characteristic derivative operators Δ_i defined by $\Delta_i\, g=Q_i$ ($i=1,2,3$). It is then straightforward to verify the following commutation relations:

$$[\Delta_1, \Delta_2]\, g = \Delta_1 Q_2 - \Delta_2 Q_1 = 0 \quad \Leftrightarrow \quad [\Delta_1, \Delta_2] = 0\ ;$$

$$[\Delta_1, \Delta_3]\, g = \Delta_1 Q_3 - \Delta_3 Q_1 = -g_x = -Q_1 = -\Delta_1 g \quad \Leftrightarrow \quad [\Delta_1, \Delta_3] = -\Delta_1\ ;$$

$$[\Delta_2 , \Delta_3]\, g = \Delta_2 Q_3 - \Delta_3 Q_2 = - g_t = - Q_2 = - \Delta_2 g \quad \Leftrightarrow \quad [\Delta_2 , \Delta_3] = - \Delta_2 \, .$$

Next, we consider the "internal" transformation (i.e., transformation in the fiber space) $g' = g\Lambda$, where Λ is a non-singular constant matrix. Then,

$$F[g'] = \Lambda^{-1} F[g]\, \Lambda = 0 \mod F[g] \, ,$$

which indicates that this transformation is a symmetry of (7.1). Setting $\Lambda = 1 + \alpha M$, where α is an infinitesimal parameter while M is a constant matrix, we write, in infinitesimal form, $\delta g = \alpha g M$. The symmetry characteristic is $Q[g] = gM$, so that $\Phi[g] = M$. Substituting for Φ into the symmetry condition (7.2), we find:

$$S(\Phi; g) = \big[F[g] \, , M \big] = 0 \mod F[g] \, .$$

Given a matrix function $g(x,t)$ satisfying the PDE (7.1), consider the following system of PDEs for two functions $\Phi[g]$ and $\Phi'[g]$:

$$\begin{aligned} \Phi'_x &= \Phi_t + [g^{-1} g_t \, , \Phi] \\ -\Phi'_t &= \Phi_x + [g^{-1} g_x \, , \Phi] \end{aligned} \qquad (7.4)$$

The *integrability condition* (or consistency condition) $(\Phi'_x)_t = (\Phi'_t)_x$ of this system requires that Φ satisfy the symmetry condition (7.2); i.e., $S(\Phi; g) = 0$. Conversely, by applying the integrability condition $(\Phi_t)_x = (\Phi_x)_t$ and by using the fact that g is a solution of $F[g] = 0$, one finds that Φ' must also satisfy (7.2); i.e., $S(\Phi'; g) = 0$.

We conclude that, for any function $g(x,t)$ satisfying the PDE (7.1), the system (7.4) is an *auto-Bäcklund transformation* (BT) [18] relating solutions Φ and Φ' of the symmetry condition (7.2) of this PDE; that is, relating different symmetries of the chiral field equation. Thus, if a symmetry characteristic $Q = g\Phi$ of the PDE (7.1) is known, a new characteristic $Q' = g\Phi'$ may be found by integrating the BT (7.4); the converse is also true. Since the BT (7.4) produces new symmetries from old ones, it may be regarded as a *recursion operator* for the PDE (7.1) [8-11,15-17].

As an example, consider the internal-symmetry characteristic $Q[g] = gM$ (where M is a constant matrix) corresponding to $\Phi[g] = M$. By integrating the BT (7.4) for Φ', we get $\Phi' = [X, M]$ and thus $Q' = g[X, M]$, where X is the "potential" of the PDE (7.1), defined by the system of PDEs

$$X_x = g^{-1} g_t \quad , \quad -X_t = g^{-1} g_x \qquad (7.5)$$

Note the *nonlocal* character of the BT-produced symmetry Q', due to the presence of the potential X. Indeed, as seen from (7.5), in order to find X one has to *integrate* the chiral field g with respect to the independent variables x and t. The above process can be continued indefinitely by repeated application of the recursion operator (7.4), leading to an infinite sequence of increasingly nonlocal symmetries.

Unfortunately, as the reader may check, no new information is furnished by the BT (7.4) in the case of coordinate symmetries (for example, by applying the BT for $Q = g_x$ we get the known symmetry $Q' = g_t$). A recursion operator of the form (7.4), however, does produce new nonlocal symmetries from coordinate symmetries in related problems with more than two independent variables, such as the self-dual Yang-Mills equation [8-11,15].

8. Concluding remarks

The algebraic approach to the symmetry problem of PDEs, presented in this article, is, in a sense, an extension to matrix-valued problems of the ideas contained in [1], in much the same way as [4] and [5] constitute a generalization of the Harrison-Estabrook geometrical approach [2] to matrix-valued (as well as vector-valued and Lie-algebra-valued) PDEs. The main advantage of the algebraic approach is the bypassing of the difficulty associated with the differential-operator representation of the symmetry-generating vector fields that act on matrix-valued functions in the jet space.

It should be noted, however, that the standard methods [1,4,5] are still most effective for *calculating* symmetries of PDEs. In this regard, one needs to enrich the ideas presented in this article by describing a systematic process for evaluating (not just recognizing) symmetries, in the spirit of [4,5]. This will be the subject of a subsequent article.

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