MULTICRITERIA OPTIMIZATION

Best Simultaneous Approximation of Functions

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Abstract. The problem we consider is to find (the) **best approximation**(s) to a given function **simultaneously** with respect to more than one criterion of proximity. Questions of existence, characterization, unicity and computation are examined. Examples are given.

Keywords: Best vectorial approximation(s), minimal projection norms, computational schemata

1. INTRODUCTION

Among other formulations of simultaneous approximation, the notion of a "Vectorially Minimal Approximation" is introduced, which is shown to be the **natural setting** for problems of simultaneity, both theoretically as well as computationally. For the above formulations of Multicriteria Optimization we propose 3 types of "models" and show their interrelationships in each "primal" and "dual" spaces. In particular, attention has been given to effective models suitable for **numerical computation.** A related problem situated in the "dual space" of approximation operators is to approximate the (non-linear) **best approximation operator** by projection operators. This approach, as a tool of "good" approximation of functions (in situations to be specified), is motivated by the following inequality, where the role of minimal projections, i.e. $min\|P\|$ is self-explanatory.

$$\|f - Pf\| \leq \|I - P\| dist(f, Y) \leq (1 + \|P\|) dist(f, Y).$$

Here again the approximation in the operator space is done **simultaneously** with respect to several norms. As just indicated, this reduces to finding "simultaneously" **minimal projection norms**. Examples are given and a "Zero in the Convex Hull" as well as a "Kolmogorov-type" characterization theorems are presented.

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The tools used in this presentation are Elementary Optimization Theory, Computational Numerical Analysis and Elementary Functional Analysis.

2. VECTORIAL APPROXIMATION

Let $\|\cdot\|_{\boldsymbol{a}}, \|\cdot\|_{\boldsymbol{b}}$ be two norms defined on a linear space $\boldsymbol{S}$ and let $\boldsymbol{f} \in \boldsymbol{S} \sim \boldsymbol{K}$ be a given function to be approximated by approximation $\boldsymbol{p} \in \boldsymbol{K} \subset \boldsymbol{S}$. $\boldsymbol{K}$ is assumed to be a closed, convex, proper subset of $\boldsymbol{S}$. Let $\boldsymbol{G}(\boldsymbol{p}) = (\,||\,\boldsymbol{f}-\boldsymbol{p}\,||_{\boldsymbol{a}}, ||\,\boldsymbol{f}-\boldsymbol{p}||_{\boldsymbol{b}})$ and define the partial ordering $\trianglerighteq$ on $\boldsymbol{G}(\boldsymbol{K})$ by

$$G(p) \trianglelefteq G(q) \Leftrightarrow \begin{cases} \|f-p\|_a \leq \|f-q\|_a \\ \quad\quad and \\ \|f-p\|_b \leq \|f-q\|_b \end{cases}$$

We shall write $G(p) \lhd G(q)$ if and only if $G(p) \trianglelefteq G(q)$ and $G(p) \neq G(q)$.

Definition 2.1
We say that p is a best vec approximation if there does not exist a $q \in K$ such that $G(q) \lhd G(p)$.

Definition 2.2
The minimal set M is given by

$$M = \{G(p) \colon p \in K \; is\; a\; best\; vec\; approximation\}.$$

There are some general geometric facts that are easy to verify. We cite some of them here:

- $\pi_1\big(G(K)\big)$ has zero homotopy group.
- M is a convex, decreasing arc.

Let Λ is the $45°$ bisector of the $\|\cdot\|_{\boldsymbol{a}}, \|\cdot\|_{\boldsymbol{b}}$ orthogonal axes. L is the supporting line to $G(K)$ which makes $135°$ angle with $\|\cdot\|_{\boldsymbol{a}}$ axes.
The proof of the following theorem is a consequence of the definitions, convexity and, in the case of $(P_m) = M \cup L$, the continuity of the best approximation operator. Sum here denotes the sum of two norms. Max means the maximum of two norms.

Theorem 2.1
Let p_s be a best sum approximation. Then $G(P_s) \subseteq M \cup L$. Similarly, if p_m denotes the best max approximation then $G(P_m) = M \cup L$ (assuming $M \cup L \neq \emptyset$).

Furthermore, we define the set D by

$$D = \left\{d \colon \inf_{q\in K}\|f-q\|_a \leq d \leq \inf_{q\in B}\|f-q\|_a\right\}$$

where,

$$B = \left\{r \in K \colon \|f-r\|_b = \inf_{q\in K}\|f-q\|_b\right\}.$$

Theorem 2.2
An element $p \in K$ is a best vectorial approximation if and only if there exists $d \in D$ and $\Phi \in S^*$ satisfying

$$\|\Phi\|_b = 1$$
$$\Phi(f-b) = \|f-q\|_a \leq d$$

and

$$Re\Phi(p-q) \leq 0 \text{ for all } q \in K \text{ satisfying } \|f-q\|_a \leq d.$$

3. VECTORIALY MINIMAL PROJECTIONS

Let $\Lambda = \Lambda(X, V)$ be the space of all linear operators from a real or complex space X into a finite-dimensional subspace V, and let Π be the family of all operators in Λ with a given fixed action on V (e.g., the identity action corresponds to the family of projections onto V). Let X be equipped with norms $\|\cdot\|_i, i = 1,2,\dots,k$. Let X_i denote the normed space given by X with the norm $\|\cdot\|_i$, and define

$$\|x\| := (\|x\|_1, \|x\|_2, \dots, \|x\|_k).$$

Define the partial ordering " $\trianglelefteq$ " on X by

$$\|x\| \trianglelefteq \|z\| \Leftrightarrow \|x\|_i \leq \|z\|_i \; for\ every\ i = 1,2,\dots,k.$$

We write $\|x\| \vartriangleleft \|z\|$ if and only if $\|x\| \trianglelefteq \|z\|$ and $\|x\| \neq \|z\|$.

Definition 3.1
For $Q \in \Lambda$, let $\|Q\|_i$ denote the operator norm on X_i, let $\|Q\| := \left(\|Q\|_1, \|Q\|_2, \dots, \|Q\|_k\right)$ and define the partial ordering " $\trianglelefteq$ " on Λ by

$$\|P\| \trianglelefteq \|Q\| \Leftrightarrow \|P\|_i \leq \|Q\|_i \; for\ every\ i = 1,2,\dots,k.$$

We write $\|P\| \vartriangleleft \|Q\|$ if and only if $\|P\| \trianglelefteq \|Q\|$ and $\|P\| \neq \|Q\|$.
P is a vectorially minimal operator in Π if there no exist $Q \in \Pi$ such that $\|Q\| \vartriangleleft \|P\|$.

Notation
The minimal set M is given by

$$M \coloneqq \{\|P\|: P \in \Pi \; is\ a\ vectorially\ minimal\ operator\ in\ \Pi\}.$$

Definition 3.2
For $i = 1,2,\dots,k$ $(x,y) \in S({X_i}^{**}) \times S({X_i}^{*})$ will be called an extremal pair for $Q \in \Lambda$, if $\langle Q_i^{**}x, y\rangle = \|Q\|_i$, where $Q_i^{**}: {X_i}^{**} \longrightarrow V$ is the second adjoint extension of Q to ${X_i}^{**}$.
(S denotes the unit sphere).

Notation
Let $E(Q)$ be the set of all extremal pairs for Q. To each $(x,y) \in Q$ associate the rank-one operator $y \otimes x$ from X_i to ${X_i}^{**}$ given by $(y \otimes x)(z) = \langle z, y\rangle x$ for $\in X_i$, where i is the subscript associated with (x, y).

Theorem 3.1 (Characterization)
P has vectorially minimal norm in Π **if and only if** the closed convex hull of $\{y \otimes x\}_{(x,y)\in E(P)}$ contains an operator E_P for which V is an invariant subspace, i.e.

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$$\boldsymbol{E_P} = \int_{\boldsymbol{E(P)}} y \otimes x d\mu(x,y) : V \longrightarrow V.$$

Theorem 3.2
P has vectorially minimal norm in Π **if and only if** there does not exist $D \in \Delta = \{D \in \Lambda : D = 0 \ in \ V\}$ such that

$$\sup_{(x,y)\in E(P)} Re\langle P_i^{**}x, y\rangle \overline{\langle D^{**}x, y\rangle} < 0.$$

4. SOME SPECIAL CASES

We give some examples of **Theorem 2.2**. In the notation of this theorem, let
$S = C[a,b]$,
$K = \Pi_n[a,b]$ (the set of polynomials on $[a,b]$ of degree less than or equal to n), $\|\cdot\|$ is the supremum norm on $[a,b]$ and $w_1, w_2 \in C[a,b]$ two (weight) functions, positive and continuous on $[a,b]$.
We introduce extreme points, for a given $f \in C[a,b]$ to be approximated, in connection with the next theorem, as follows:

$$\underline{\overline{X}}_{+1} = \{x \in [a,b] : w_1(x)\big(f(x) - p(x)\big) = +\|w_1(f-p)\|\}$$
$$\underline{\overline{X}}_{+2} = \{x \in [a,b] : w_2(x)\big(f(x) - p(x)\big) = +\|w_2(f-p)\|\}$$
$$\underline{\overline{X}}_{-1} = \{x \in [a,b] : w_1(x)\big(f(x) - p(x)\big) = -\|w_1(f-p)\|\}$$
$$\underline{\overline{X}}_{-2} = \{x \in [a,b] : w_2(x)\big(f(x) - p(x)\big) = -\|w_2(f-p)\|\}.$$

$$\underline{\overline{X}}_p = \underline{\overline{X}}_{+1} \cup \underline{\overline{X}}_{+2} \cup \underline{\overline{X}}_{-1} \cup \underline{\overline{X}}_{-2}$$

The sign function $\sigma(x)$ on $\underline{\overline{X}}_p$ is defined by

$$\sigma(x) = -1 \ when \ x \in \underline{\overline{X}}_{-1} \cup \underline{\overline{X}}_{-2}$$

and

$$\sigma(x) = +1 \ when \ x \in \underline{\overline{X}}_{+1} \cup \underline{\overline{X}}_{+2}.$$

Theorem 4.1 (Application)
Consider the Vectorial Chebyshev optimization, with w_1 and w_2 as defined above. Then p is a best vec approximation to f if and only if there exist $n+2$ points $x_1 < x_2 < \cdots < x_{n+2} \in \underline{\overline{X}}_p \subset [a,b]$ satisfying

$$\sigma(x_i) = (-1)^{i+1}\sigma(x_1) \ \ for \ every \ i = 1,2,\dots,n+2.$$

Theorem 4.2
Each best vec approximation is unique; i.e. given $\mu \in M$ there is only one $p \in \Pi_n[a,b]$ such that
$G(p) = \mu$.

Note that this uniqueness does not contradict the fact that the minimal set M has, in general, an infinite number of points, each of which corresponds to a (unique) best vectorial approximation. Likewise, the easily shown existence of M proves the existence of best solutions.

Theorem 4.3 (Application)
Let $X = C[a,b], K = \Pi_n[a,b], \|\cdot\|_a, \|\cdot\|_b$ the sup and L_2 norms on $C[a,b]$ which we denote by $\|\cdot\|_\infty$ and $\|\cdot\|_2$ respectively.
Find the best vectorial approximation p_d whose error in Chebyshev norm equals a prescribed value $d \in P^+$, $\| f - p_1 \|_\infty \leq d \leq \| f - p_2 \|_\infty$. It is clear that the desired polynomial p_d is the unique solution to the problem

$$\min_{p \in \Pi_n} \|f - p\|_2$$

subject to

$$\| f - p\|_\infty \leq d.$$

Since the number of constraints here is infinite, we proceed by solving a sequence of quadratic programming problems, each with a finite number of constraints. The sequence of solutions $\{p_k\}$ is shown to converge to the theoretical solution p_d.

Algorithm Corresponding to Theorem 4.3
At the $k-th$ step we have from the preceding steps a finite set of points $X^k \subset [a,b]$. We solve the quadratic program

$$\min_{p \in \Pi_n} \|f - p\|_2$$

subject to

$$\| f(x) - p(x)\|_\infty \leq d, x \in X_k.$$

Denoting by p_k the solution of this problem, we calculate a point $x_k \in [a,b]$ such that

$$|f(x_k) - p_k(x_k)| = \| f - p_k \|_\infty.$$

We form $X^{k+1} = X^k \cap \{x_k\}$ and proceed to the next cycle. At the beginning X^1 may be an arbitrary finite set, containing a maximum of $|f(x) - p_L(x)|$.

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